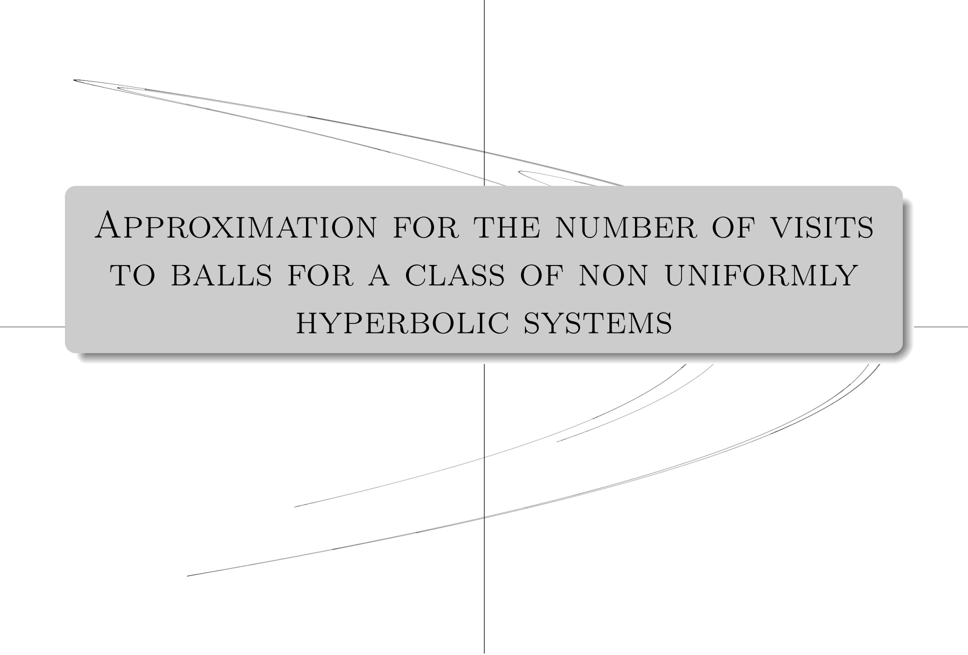
Approximation for the number of visits to balls for a class of non uniformly hyperbolic systems



Joint work with Pierre COLLET:

Poisson approximation for the number of visits to balls in non-uniformly hyperbolic dynamical systems.

Ergodic Theory & Dynamical Systems **33**, 49-80 (2013).

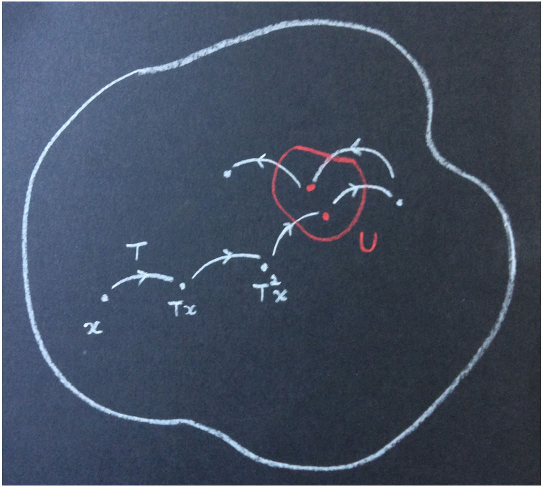
$T : M \circlearrowleft$ a diffeo acting on a compact Riemannian manifold M & such that $\mu \circ T^{-1} = \mu$ for an ergodic probability measure μ.

PICK:

- $x \in M$
- $U \subset M$
- $t > 0$

THEN COUNT:

$$\sum_{j=0}^{\lfloor t/\mu(U) \rfloor} \mathbb{1}_{U}(T^j x)$$

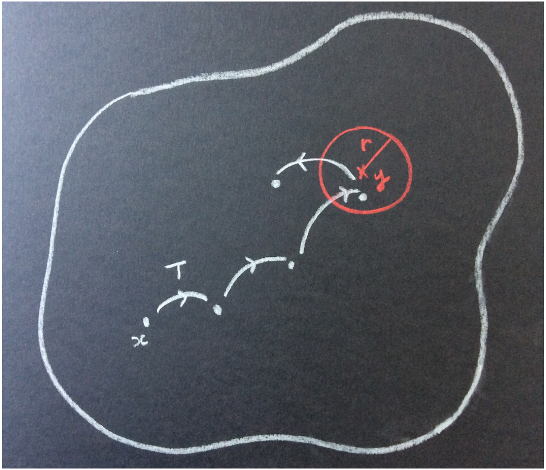


PICK:

$U = B_r(y)$

THEN COUNT:

$$\sum_{j=0}^{\lfloor t/\mu(B_r(y))\rfloor} \mathbb{1}_{B_r(y)}(T^j x)$$



Basic question

(M, T, μ) with 'enough mixing'

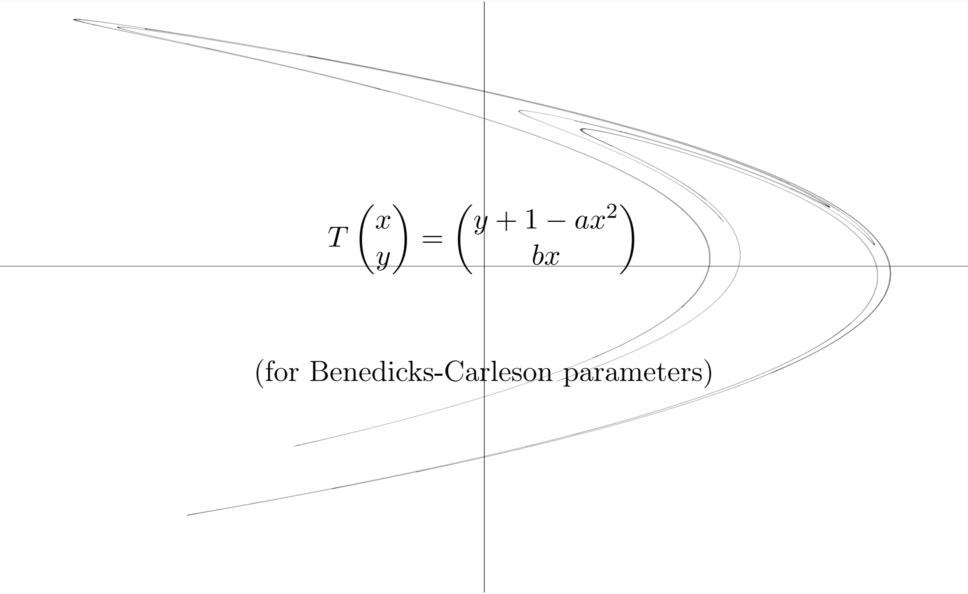
$+\ y$ 'typical'

$+\ r$ small enough

$$\overset{?}{\Longrightarrow}$$

$$\sum_{j=0}^{\lfloor t/\mu(B_r(y)) \rfloor} \mathbb{1}_{B_r(y)} \circ T^j \overset{\text{law}}{\approx} \text{Poisson}(t)$$

Our motivating example: the Hénon map



$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y + 1 - ax^2 \\ bx \end{pmatrix}$$

(for Benedicks-Carleson parameters)

Abstract setting

Nonuniformly hyperbolic systems modeled by Young towers (Annals of Math., 1998)

$$+$$

Extra assumption: **1D local unstable manifolds**.

Young tells us that there exists a SRB measure μ and exponential decay of correlations for Hölder observables.

Theorem (J.-R. C. – P. Collet)

There exist constants $C, a, b > 0$ such that for all $r \in (0,1)$:

- There exists a set $\mathcal{M}_r$ such that

$$\mu(\mathcal{M}_r) \le C r^b\,;$$

- For all $y \notin \mathcal{M}_r$ one has

$$\left| \mu \left\{ x \in M \,\middle|\, \sum_{j=0}^{\lfloor t/\mu(B_r(y)) \rfloor} \mathbb{1}_{B_r(y)}(T^j x) = k \right\} - \frac{t^k}{k!}\, e^{-t} \right| \le C\, r^a$$

  for every integer $k \ge 0$ and for every $t > 0$.

Let $(X_n)_{n\in\mathbb{N}}$ be a stationary $\{0,1\}$-valued process.
Set $\varepsilon := \mathbb{P}(X_1 = 1)$.

Then, for all positive integers p, M, N such that $M \leq N-1$ and $2 \leq p < N$, one has

$$d_{\mathrm{TV}}\big(X_1 + \cdots + X_N, \mathrm{Poisson}(N\varepsilon)\big) \leq R(\varepsilon, N, p, M)$$

where

$$R(\varepsilon, N, p, M) = 2NM\big(R_1(\varepsilon, N, p) + R_2(\varepsilon, p)\big) + R_3(\varepsilon, N, p, M).$$

Application to our dynamical systems

$$X_j = \mathbb{1}_{B_r(y)} \circ T^{j-1}$$

$$\varepsilon = \mu(B_r(y))$$

$$N = \lfloor t/\mu(B_r(y)) \rfloor$$

$$p = \mathcal{O}(1) \lfloor \log(r^{-1}) \rfloor$$

$$M = 1 + \lfloor \log(r^{-1}) \rfloor$$

About R_3

$$R_3(\varepsilon, N, p, M) = 4\left(Mp\varepsilon(1+N\varepsilon) + \frac{(\varepsilon N)^M}{M!}\ e^{-N\varepsilon} + N\varepsilon^2\right)$$

$$R_2(\varepsilon, p) = \sum_{\ell=1}^{p-1} \mathbb{E}\left(\mathbb{1}_{\{X_1=1\}}\, \mathbb{1}_{\{X_{\ell+1}=1\}}\right)$$

Since $X_j = \mathbb{1}_{B_r(y)} \circ T^{j-1}$ one has

$$\mathbb{1}_{\{X_1=1\}}\mathbb{1}_{\{X_{\ell+1}=1\}} = 1 \quad \text{if} \quad B_r(y) \cap T^{\ell}(B_r(y)) \neq \emptyset.$$

For ℓ 'small' this means that y recurs 'fast'. We prove that

$$\mu\left\{y : \exists 1 \leq k \leq \mathfrak{c}\lfloor \log(r^{-1}) \rfloor : B_r(y) \cap T^{\ell}(B_r(y)) \neq \emptyset\right\} \leq \mathcal{O}(1)\, r^{b'}.$$

For ℓ 'large', use decay of correlations.

About R_1

We need to get

$$\left| \int \mathbb{1}_{B_r(y)}(x)\ \mathbb{1}_{B_r(y)}(T^\ell x)\ \mathrm{d}\mu(x) - \big(\mu(B_r(y))\big)^2 \right| \le \theta(r)\ \mu(B_r(y))$$

with $\lim_{r\to 0} \theta(r) = 0$ except for y in set whose μ-measure goes to 0 when $r \to 0$.

PROBLEM:
we must replace the $\mathbb{1}_{\text{ball}}$ by some Lipschitz functions.

This creates an extra error given by the measure of the coronas

$$B_r(y) \backslash B_{r-r^\delta}(y) \quad (\delta > 1).$$

Again, we can exclude a set of 'bad' centers outside which

$$\mu\big(B_r(y) \backslash B_{r-r^\delta}(y)\big) = o(\mu(B_r(y)))$$

provided δ is large enough (exponential decay of correlations needed for that).