Saturable Schrödinger equations and systems: Existence and related topics

Benedetta Pellacci
Università di Napoli “Parthenope”

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Before Starting

Joint research with:

Liliane **Maia** (University of Brasilia)

Eugenio **Montefusco** (*Sapienza* università di Roma)

★ *"Weakly coupled nonlinear Schrödinger systems: the saturation effect."*
[Calc.Var.PDE 2013] with Liliane Maia and Eugenio Montefusco

♣ *"Singularly perturbed elliptic problems with nonautonomous asymptotically linear nonlinearities."*
Nonlinear Analysis TMA (2015) with Liliane Maia and Eugenio Montefusco

★ *"Positive solutions for asymptotically linear problems in exterior domains."* Work in Progress. with Liliane Maia

Photorefractive Crystals

★ In the study of propagation of light beams in Kerr media we have to take into account *the birefrengence effect*; while when we take a photorefractive crystals we can see *the saturable effect*. The absorption of light decreases when increasing light intensity. As a consequence, at a certain threshold absorption of light saturates.

★ The model of the classical well-known cubic Schrödinger equation is substituted by an asymptotically linear one.

References: [Litchinister,Krolikowski, Akhmediev Agrawal *Phys.Rev.E* (1999)], [Gatz, Herrmann *J.Opt.Soc.* (1997)], [Ostrovskaya, Kivshar *J.Opt.B.* (1999)], [Weilnau, Ahles, Petter *Ann.der Phys.* (2002)].

Saturable Equation

$$i\frac{\partial \Phi}{\partial t} + \Delta\Phi + \frac{|\Phi|^2}{1+s|\Phi|^2}\Phi = 0 \quad \text{in } \mathbb{R}^N$$

- ★ $N \geq 2$ (differently from the cubic case) ,
- ★ Φ denotes the amplitude of the beam,
- ★ s is the saturation parameter.
- ★ Looking for $\Phi(x,t) = u(x)e^{i\lambda t}$ we end up with

$$\begin{cases} -\Delta u + \lambda u = \dfrac{u^3}{1+su^2} & \text{in } \mathbb{R}^N. \\ u(x) \to 0 \quad \text{as } |x| \to \infty. \end{cases}$$

The action functional is given by

$$J_{\lambda,s}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2}\left(\lambda - \frac{1}{s}\right)\int_{\mathbb{R}^N} u^2 + \frac{1}{2s^2}\int_{\mathbb{R}^N} \ln\left(1+su^2\right)$$

A simple observation

$$\begin{cases} -\Delta u + \lambda u = \dfrac{u^3}{1+su^2} & \text{in } \mathbb{R}^N. \\ u(x) \to 0 \quad \text{as } |x| \to \infty. \end{cases}$$

- Recall Pohozaev identity:

$$0 \leq (N-2)\int_{\mathbb{R}^N} |\nabla u|^2 = N\left\{\left(\frac{1}{s} - \lambda\right)\int_{\mathbb{R}^N} u^2 - \frac{1}{2s^2}\int_{\mathbb{R}^N} \lg\left(1+su^2\right)\right\}$$

Remark

For $\lambda \geq 1/s$ the unique solution has to be $u \equiv 0$.
In particular, there are no positive solution for the problem

$$\begin{cases} -\Delta u + u = \dfrac{u^3}{1+u^2} & \text{in } \mathbb{R}^N. \\ u(x) \to 0 \quad \text{as } |x| \to \infty. \end{cases}$$

The Single Autonomous Equation

$$\begin{cases} -\Delta u + \lambda u = \dfrac{u^3}{1 + s u^2} & \text{in } \mathbb{R}^N. \\ u(x) \to 0 \quad \text{as } |x| \to \infty. \end{cases}$$

- Existence:

  ★ $N \geq 3$ [Berestycki-Lions, 1983],

  ★ [Stuart-Zhou, 1999];

  ★ $N = 2$ [Berestycki-Gallouët-Kavian, 1984] **There exists a positive, regular, radially symmetric solution iff** $\lambda s < 1$.

- Uniqueness and simmetry:

  ♣ [Serrin-Tang, 2000, Serrin-Zou 1999]

The Model System

$$\begin{cases} -\Delta u + \lambda u = \dfrac{u(u^2+v^2)}{1+s(u^2+v^2)} & \text{in } \mathbb{R}^N \\ -\Delta v + \lambda v = \dfrac{v(u^2+v^2)}{1+s(u^2+v^2)} & \text{in } \mathbb{R}^N, \end{cases}$$

Theorem

The unique-up to rotation-solution U with both positive components of the model problem is given by $U = U_\theta = (u,v) = (\cos\theta, \sin\theta) z_\lambda$, for $\theta \in (0, \pi/2)$ and z_λ the unique positive solution of

$$\begin{cases} -\Delta z_\lambda + \lambda z_\lambda = \dfrac{z_\lambda^3}{1+s z_\lambda^2} & \text{in } \mathbb{R}^N. \\ z_\lambda(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

A General Problem I

$$\begin{cases} -\Delta u + \lambda_1 u = \dfrac{\alpha u(\alpha u^2 + \beta v^2)}{1 + s(\alpha u^2 + \beta v^2)} & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \dfrac{\beta v(\alpha u^2 + \beta v^2)}{1 + s(\alpha u^2 + \beta v^2)} & \text{in } \mathbb{R}^N, \end{cases}$$

Definition

A vectorial solution is a solution with both nontrivial components. While, a scalar-or semitrivial-solution is a solution with one trivial component.

★ This model allows a vector $U = (u, v)$ to split its L^2 norm between the components not with equal weights.

★ Why this choice of constants? We want the problem to be variational!

★ For $s = 0$ we have a system of two weakly coupled cubic Schrödinger equations with coupling coefficient given by $\alpha\beta$

Preliminary Results

$$\begin{cases} -\Delta u + \lambda_1 u = \dfrac{\alpha u(\alpha u^2 + \beta v^2)}{1 + s(\alpha u^2 + \beta v^2)} & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \dfrac{\beta v(\alpha u^2 + \beta v^2)}{1 + s(\alpha u^2 + \beta v^2)} & \text{in } \mathbb{R}^N, \end{cases}$$

$$I(U) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 + (\lambda_1 - \alpha/s)u^2 + (\lambda_2 - \beta/s)v^2 + 1/(2s^2)\int_{\mathbb{R}^N} \ln(1 + s(\alpha u^2 + \beta v^2))$$

★ $s > max\{\alpha/\lambda_1, \beta/\lambda_2\} \Rightarrow U \equiv (0,0)$ (Pohozaev)

★ [Brezis-Lieb, 1984]: $\exists U \not\equiv (0,0)$ least action solution, via constrained minimization methods.

Theorem

If $s < max\{\alpha/\lambda_1, \beta/\lambda_2\}$, then, there exists a least action solution $U \neq (0,0)$.

Necessary conditions

$$\begin{cases} -\Delta u + \lambda_1 u = \dfrac{\alpha u(\alpha u^2 + \beta v^2)}{1 + s(\alpha u^2 + \beta v^2)} & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \dfrac{\beta v(\alpha u^2 + \beta v^2)}{1 + s(\alpha u^2 + \beta v^2)} & \text{in } \mathbb{R}^N, \end{cases}$$

Theorem

If there exists a vectorial solution $U = (u, v)$ with $u, v > 0$, then the following equivalence holds:

$$\lambda_2 < \lambda_1 \quad \Leftrightarrow \quad \beta < \alpha \quad \Leftrightarrow \quad \frac{\beta}{\lambda_2} > \frac{\alpha}{\lambda_1}$$

$$\lambda_2 = \lambda_1 \quad \Leftrightarrow \quad \alpha = \beta.$$

If $\alpha = \beta$ and $\lambda_1 = \lambda_2 = \lambda$, then the vectorial solution is $U = 1/\alpha(\cos\theta, \sin\theta) z_\lambda$, with z_λ the solution of the single equation.

Assume: $\lambda_2 < \lambda_1$, $\beta < \alpha$, $\beta/\lambda_2 > \alpha/\lambda_1$.

- $(z_\alpha, 0), (0, z_\beta)$ come from z_α, z_β solutions of

$$-\Delta z_\alpha + \lambda_1 z_\alpha = \frac{\alpha^2 z_\alpha^3}{1 + s\alpha z_\alpha^2}, \qquad -\Delta z_\beta + \lambda_2 z_\beta = \frac{\beta^2 z_\beta^3}{1 + s\beta z_\beta^2}$$

- $z_\alpha = 1/\sqrt{\alpha s}\,\varphi_\alpha(\sqrt{\lambda_1}x)$, $\qquad z_\beta = 1/\sqrt{\beta s}\,\varphi_\beta(\sqrt{\lambda_2}x)$

- $-\Delta\varphi_\alpha + \varphi_\alpha = \frac{\alpha}{s\lambda_1}\frac{\varphi_\alpha^3}{1+\varphi_\alpha^2}$, $\qquad -\Delta\varphi_\beta + \varphi_\beta = \frac{\beta}{s\lambda_2}\frac{\varphi_\beta^3}{1+\varphi_\beta^2}$

- $I_\beta(\varphi_\beta) < I_\alpha(\varphi_\alpha)$, (pink condition)

- $I(z_\alpha, 0) = \frac{\lambda_1^{1-N/2}}{s\alpha} I_\alpha(\varphi_\alpha) > \frac{\lambda_1^{1-N/2}}{s\alpha} I_\beta(\varphi_\beta) = \frac{\lambda_1^{1-N/2}}{\cancel{s}\alpha}\frac{\cancel{s}\beta}{\lambda_2^{1-N/2}} I(0, z_\beta)$

- Then we obtain a comparison between $I(z_\alpha, 0)$ and $I(0, z_\beta)$ if $(\lambda_1/\lambda_2)^{1-N/2} \cdot \beta/\alpha > 1$ which contradicts the necessary conditions red and blue for $N \geq 2$.

Least action solutions are scalar !

Necessary for nontrivial solutions: $s < \beta/\lambda_2$.
Necessary for both positive components:
$\lambda_2 < \lambda_1, \quad \beta < \alpha \quad \beta/\lambda_2 > \alpha/\lambda_1$.

Theorem

If $\dfrac{\alpha}{\lambda_1} \leq s < \dfrac{\beta}{\lambda_2}$ *then the least action solution is* $(0, z_\beta)$.

$$I(U) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 + (\lambda_1 - \alpha/s)u^2 + (\lambda_2 - \beta/s)v^2$$
$$+1/(2s^2)\int_{\mathbb{R}^N} \lg(1 + s(\alpha u^2 + \beta v^2))$$

Theorem

If $\dfrac{\alpha-\beta}{\lambda_1-\lambda_2} \leq s < \dfrac{\alpha}{\lambda_1}$, *then the least action solution is* $(0, z_\beta)$.

From $U = (u, v)$ we pass to $w = \sqrt{u^2+v^2}$ and $I(U) > I_\beta(w)$.

Recent development

$$\begin{cases} -\Delta u + \lambda_1 u = \dfrac{\alpha u(\alpha u^2 + \beta v^2)}{1 + s(\alpha u^2 + \beta v^2)} & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \dfrac{\beta v(\alpha u^2 + \beta v^2)}{1 + s(\alpha u^2 + \beta v^2)} & \text{in } \mathbb{R}^N, \end{cases}$$

♣ [Mandel, arxiv 2015]

- All least action solutions are semi-trivial for every $s < (\alpha - \beta)/(\lambda_1 - \lambda_2)$ and for every dimension $N \geq 1$.
- For $N = 2, 3$ there exists solutions with both positive components emanating from $(u, v, s) = (z_\alpha, 0, s)$ if

$$\frac{\lambda_2}{\lambda_1} < \frac{\beta}{\alpha} < \left(\frac{\lambda_2}{\lambda_1}\right)^{1-N/4} \quad \text{and} \quad s < \frac{\alpha - \beta}{\lambda_1 - \lambda_2} < \frac{\alpha}{\lambda_1}$$

- Analogous result holds for $N = 1$
- Also bifurcation of $(0, k)-$nodal solutions is studied.

Open Problems

$$\begin{cases} -\Delta u + \lambda_1 u = \dfrac{\alpha u(\alpha u^2 + \beta v^2)}{1 + s(\alpha u^2 + \beta v^2)} & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \dfrac{\beta v(\alpha u^2 + \beta v^2)}{1 + s(\alpha u^2 + \beta v^2)} & \text{in } \mathbb{R}^N, \end{cases}$$

- Can we recover the set of solutions with both positive components by variational methods? (Minimization on some Nehari set?)
- Stability of the solutions with both positive components.
- Study the case $s < 0$
- Large dimension $N \geq 4$: this problem-differently from the classical cubic-is always sub-critical!

Related result: [Lehrer EJDE2013] existence result for $V(x)$ constant and $s(x)$ variable for a strongly coupled system similar to this one.

Perturbed Elliptic Problem

$$\begin{cases} \varepsilon^2 \Delta u + V(x) u^2 = \dfrac{u^3}{1 + s(x) u^2} & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

Interest

We are interested in families of solutions concentrating and developing a spike shape around one or more particular points of $\mathbb{R}^N$ and vanishing elsewhere as $\varepsilon \to 0$.

- V is Hölder continuous and $V(x) \geq \mu > 0$.
- s is Hölder continuous and $s(x) \geq \alpha > 0$.

Perturbed Elliptic Problem

$$\begin{cases} \varepsilon^2 \Delta u + V(x) u^2 = \dfrac{u^3}{1 + s(x) u^2} & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

For $s(x) \equiv 0$: many contributes based on two main approaches:

- Lyapunov-Schmidt reduction method
  Floer-Weinstein (1986), Oh (1988-1990), Ambosetti-Badiale-Cingolani (1997), Li (1997), Grossi (2000), Ambosetti-Malchiodi-Secchi (1997), Pistoia (2002), Kang-Wei (2000), and the book by Ambrosetti-Malchiodi (2005).
- Variational Methods and Penalization procedure:
  Rabinowitz (1992) Del Pino-Felmer (1996-1997-1998...) Bonheure-Van Schaftingen (2008), Byeon-Jeanjean (2007) D'Avenia-Pomponio-Ruiz (2012)
- and many others....

There is not a Fundamental Problem!

$$\begin{cases} -\Delta Q_{\lambda,\mu} + \lambda Q_{\lambda,\mu} = \dfrac{Q_{\lambda,\mu}^3}{1+\mu Q_{\lambda,\mu}^2} \\ Q_{\lambda,\mu}(x) \to 0 \quad \text{as } |x| \to \infty \end{cases}$$

★ Recall that there exists a solution iff $\lambda\mu < 1$.

♣ We would like to have $Q_{\lambda,\mu} = \lambda^{\sigma}\mu^{\nu} R(\lambda^{\sigma} x)$, with R solution of

$$\begin{cases} -\Delta R + R = \dfrac{R}{1+R^2} \\ R(x) \to 0 \quad \text{as } |x| \to \infty. \end{cases}$$

★ There are no nontrivial solutions for this problem!

♣ We cannot express $Q_{\lambda,\mu}$ as a member of a two-parameters family generating by a fundamental solution.

Known Results

$$\begin{cases} \varepsilon^2 \Delta u + V(x) u^2 = \dfrac{u^3}{1 + s(x) u^2} & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

[Jeanjean-Tanaka 2004]

- It is studied the case $s(x) \equiv s$, with general asymptotically linear non-linearities $f(u)$.
- General hypotheses: $f(u)/u$ is not assumed to be increasing. (this hypothesis is satisfied in our case)
- Concentration around minimum points of the potential $V(x)$.

[Wang-Xu-Zhang 2009]

- V is unbounded from above and may change sign;
  s is bounded from above. Existence results for $\varepsilon > 0$ are proved via a linking argument.
- The concentration is not studied.

Locating the possible concentration points

$$\begin{cases} \Delta u + V(z)u^2 = \dfrac{u^3}{1+s(z)u^2} & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

- For every $z \in \mathbb{R}^N$, consider the frozen functional $I_z : H^1 \to \mathbb{R}$ defined by

$$I_z(u) = \frac{1}{2}\|\nabla u\|^2 + \left(V(z) - \frac{1}{s(z)}\right)\|u\|^2 + \frac{1}{2s^2(z)}\int_{\mathbb{R}^N} \lg(1+s(z)u^2).$$

- We have a positive least action solution if and only if z belongs to the open subset A of $\mathbb{R}^N$

$$\boxed{A = \{z \in \mathbb{R}^N : s(z)V(z) < 1\}}$$

Theorem

Let $z : s(z)V(z) < 1$ *and* $r > 0$ *such that*

$$\begin{cases} V(z) = \min_{B(z,r)} V(x) \leq \min_{\partial B(z,r)} V(x) \textbf{ and } s(z) = \min_{B(z,r)} s(x) < \min_{\partial B(z,r)} s(x), \\ \textbf{or} \\ V(z) = \min_{B(z,r)} V(x) < \min_{\partial B(z,r)} V(x) \textbf{ and } s(z) = \min_{B(z,r)} s(x) \leq \min_{\partial B(z,r)} s(x). \end{cases}$$

$\exists \varepsilon_0 > 0$ *such that,* $\forall \varepsilon \in (0, \varepsilon_0)$, $\exists u_\varepsilon \geq 0$, *solution of*

$$\begin{cases} \varepsilon^2 \Delta u + V(x) u^2 = \dfrac{u^3}{1 + s(x) u^2} & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

and such that:

(i) u_ε *admits exactly one global maximum point* $x_\varepsilon \in B(z,r)$*;*

(ii) $\lim_{\varepsilon \to 0} V(x_\varepsilon) = V(z)$ *and* $\lim_{\varepsilon \to 0} s(x_\varepsilon) = s(z)$;

(iii) $\exists \mu_1, \mu_2 > 0$ *such that,* $\forall x \in \mathbb{R}^N$, $u_\varepsilon(x) \leq \mu_1 e^{-\mu_2 \frac{|x - x_\varepsilon|}{\varepsilon}}$.

Remarks

Main hypothesis on V and s

$$\begin{cases} V(z) = \min_{B(z,r)} V(x) \leq \min_{\partial B(z,r)} V(x), \; s(z) = \min_{B(z,r)} s(x) < \min_{\partial B(z,r)} s(x), \\ V(z) = \min_{B(z,r)} V(x) < \min_{\partial B(z,r)} V(x), \; s(z) = \min_{B(z,r)} s(x) \leq \min_{\partial B(z,r)} s(x), \end{cases}$$

- It is not restrictive to assume that the minimum is in the centre of the ball: If $s_0 = s(z_1)$, and $V_0 = V(z_1)$ with $z_1 \in B(z,r)$, but $z_1 \neq z$, z_1 has to be in A as

$$s(z_1)V(z_1) \leq s(z)V(z) < 1,$$

  so that we can replace z with z_1, obtaining concentration around z_1.
- The strict inequality is needed only on s or V not on both, so that one between V or s may be constant.

Corollary

Corollary

Assume that

$$V \equiv V_0 \in \mathbb{R}^+.$$

Then we have concentration around local minimum points z of the function s such that $s(z) < 1/V_0$.

$$I_z(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + |u^2|) - \int_{\mathbb{R}^N} F_z(u)$$

$$F_z(u) = \frac{u^2}{2s(z)} - \frac{1}{s^2(z)} \ln(1 + s(z)u)$$

Observe that the function

$$G_u(s) = \frac{u^2}{2s} - \frac{1}{s^2} \ln(1 + su)$$

is decreasing with respect to s.

Open Problems

- Can we derive simpler and more concrete necessary conditions?
- Can we have concentration in points which are minimum points of neither V nor s?
- There is a unique function of V and s which plays the crucial role in locating the concentration points?
- Maybe one can start studying the concentration for the problem

$$\begin{cases} \varepsilon^2 \Delta u + u^2 = \dfrac{1}{V(x)s(x)} \dfrac{u^3}{1+u^2} & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

- Concentration at higher energy level, more general nonlinearities...
- Concentration for the systems case.

Why look for asymmetric solutions?

"While it is commonly believed that asymmetric solitary waves possess a higher energy, and should be a priori unstable, our results demonstrate that the opposite is true: An excited state with an elaborate geometry may indeed be more stable than a radially symmetric one and, as such, would be a better candidate for experimental realization. "
[Garcia-Ripoll, Pérez-Garcia, Ostrovskaya, Kivshar "Dipole-Mode vector soliton" Physical Review Letters (2000)]

"Consider a class of partial differential equations, invariant under a symmetry group. As the L^2 norm increases the dynamically stable state of the system is a state which is no longer invariant. That is, symmetry is broken and there is an exchange of stability".
[Kirr, Kevrekidis, Shlizerman, Weinstein "Symmetey-breaking bifurcation in nonlinear Schrödinger/Gross-Pitaevskii equations" Siam J. Math. Anal. 2008]

Saturable problems in exterior domains

$$\begin{cases} -\Delta u + \lambda u = \dfrac{u^3}{1+su^2}, & \text{in} \quad \Omega, \\ \quad u \in H_0^1(\Omega) \end{cases}$$

where Ω is an unbounded domain in $\mathbb{R}^N$, $N \geq 3$, with smooth boundary $\partial\Omega \neq \emptyset$ bounded, and such that $\mathbb{R}^N \setminus \Omega$ is bounded.

- [Li-Zheng (2006)] convex asymptotically linear non-linearity; model example: $f(u) = \dfrac{u^2}{1+su}$.

In the superlinear case

- Benci-Cerami (1987), Cerami-Passaseo (1992-1995), Bartsch-Weth (2005).
- Bahri-Li (1990) Bahri-Lions (1997), Bartsch-Willem (1993), Lorca-Ubilla (2004), Bartsch-Weth (1993), Clapp-Salazar (2006), Cerami, (2006), Ambrosetti-Cerami-Ruiz (2008)....

Our Result

Let Ω be an unbounded domain in $\mathbb{R}^N$, $N \geq 3$, with smooth boundary $\partial\Omega \neq \emptyset$ bounded, and such that $\mathbb{R}^N \setminus \Omega$ is bounded.

Theorem

Let $\lambda > 0$. There exists at least a positive solution of

$$\begin{cases} -\Delta u + \lambda u = \dfrac{u^3}{1 + su^2}, \quad in \quad \Omega, \\ \\ u \in H_0^1(\Omega) \end{cases}$$

- ♣ We handle more general asymptotically linear non-linearity
- ♣ No assumption on the size of $\mathbb{R}^N \setminus \Omega$ is supposed.

Argument in the pure power case

1. There are no least action solutions, since this would lead to the existence of a least action solution of the problem in $\mathbb{R}^N$ with compact support.

2. Careful analysis of a general Cerami sequence (even not minimizing) by proving the splitting lemma: Let m_λ be the least action level of the problem in the whole $\mathbb{R}^N$. Then, we have compactness property in the interval $(m_\lambda, 2^{1-2/p} m_\lambda)$.

4. Working in subsets of the L^p sphere and imposing additional conditions by means of a barycenter function, one can construct a linking geometry.

5. Then main point is to show that we are exactly in the interval where we have compactness.

6. This is done by a deep knowledge of the asymptotic behaviour of the least action solution of the problem in $\mathbb{R}^N$.

Main novelties

3. Let m_λ be the least action level of the problem in the whole $\mathbb{R}^N$. Then, we have compactness property in the interval $(m_\lambda, 2m_\lambda)$.

4. Working in subsets of the Nehari manifolds and imposing additional conditions by means of a barycenter function, one can construct a linking geometry.

5. Then main point is to show that we are exactly in the interval where we have compactness.

6. This is done by a deep knowledge of the asymptotic behaviour of the least action solution of the problem in $\mathbb{R}^N$, without using any homogeneity properties !

Open Problems

★ $\Omega = \mathbb{R}^N$ with asymmetric potentials.

★ Qualitative properties of the solution: what about its Morse index?

★ Is this problem related to the existence of changing sign solutions of the problem in $\mathbb{R}^N$ with small Morse index?

★ Connections with stability issue.


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Thanks!