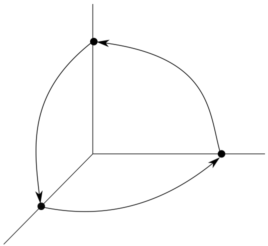
Existence and stability of heteroclinic networks in $\mathbb{R}^4$

Alexander Lohse



Joint work with Sofia Castro

AMS-EMS-SPM Meeting – Porto, 13th June 2015

Definition 1 (Krupa&Melbourne [3])

(1) A heteroclinic cycle is called **robust** if for all j there is a subgroup $\Sigma_j \subset \Gamma$ such that ξ_{j+1} is a sink in $P_j := \mathrm{Fix}(\Sigma_j)$ and $W^u(\xi_j) \cap P_j \subset W^s(\xi_{j+1})$.

(2) A robust cycle in $\mathbb{R}^4$ is called **simple** if
- $\dim(P_j) = 2$ for all j,
- it intersects connected components of $(P_{j-1} \cap P_j) \setminus \{0\}$ at most once,
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Definition 2 (Krupa&Melbourne [3])

A simple heteroclinic cycle $X \subset \mathbb{R}^4$ is of
- **type A** if and only if $\Sigma_j \cong \mathbb{Z}_2$ for all j (i.e. there is no reflection in Γ),
- **type B** if and only if X lies in a 3d fixed-point subspace $Q \subset \mathbb{R}^4$,
- **type C** if and only if it is not of type A or B.

Definition 3 (L.&Castro [5])

A heteroclinic network in $\mathbb{R}^4$ is called **simple** if

- all of its subcycles are simple,
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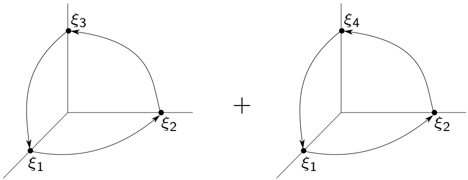

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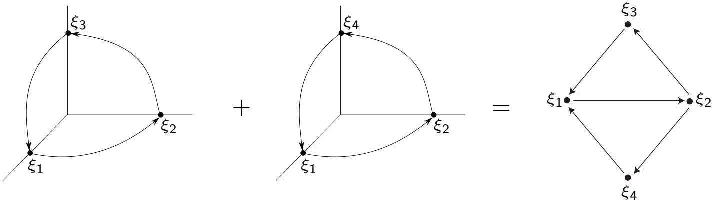

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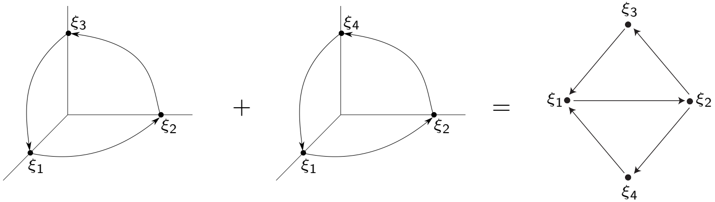

→ Can we give a list of all simple networks in $\mathbb{R}^4$?

→ What about relative stability/competition between cycles?

→ How do dynamics near these networks depend on their type?

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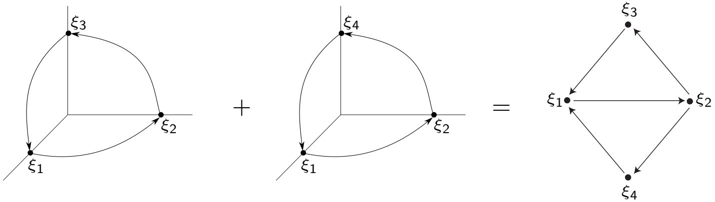


Theorem 4 (Castro&L. [5, 2])

In $\mathbb{R}^4$, the following is the complete list of (genuinely heteroclinic) simple networks that can be generated with the **simplex** *and* **cylinder methods** *from Ashwin&Postlethwaite [1]:*

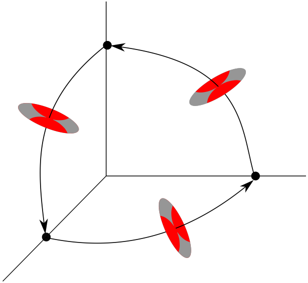
- (A_2, A_2), (A_3, A_3), (A_3, A_4), (A_3, A_3, A_4)
- (B_2^+, B_2^+), (B_3^-, B_3^-), (B_3^-, C_4^-), (B_3^-, B_3^-, C_4^-)

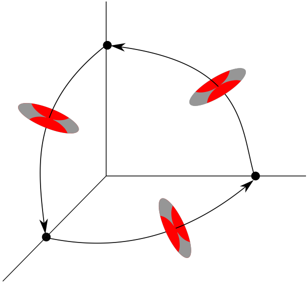
Non-asymptotic stability

Definition 5 (Melbourne [6])

A compact invariant set $X \subset \mathbb{R}^n$ is called **essentially asymptotically stable (e.a.s.)** if it is asymptotically stable relative to a set $N \subset \mathbb{R}^n$ and

$$\frac{\ell(B_\varepsilon(X) \cap N)}{\ell(B_\varepsilon(X))} \xrightarrow{\varepsilon \to 0} 1.$$





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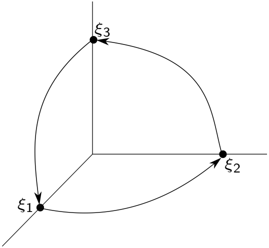
Theorem 6 (L. [4])

Let $X \subset \mathbb{R}^n$ be a heteroclinic cycle with $\ell_1(X) < \infty$. Assume that the local stability index $\sigma_{\mathrm{loc}}(x)$ exists for all $x \in X$. Then the following holds:

$$X \text{ is e.a.s.} \;\Leftrightarrow\; \sigma_{\mathrm{loc}}(x) > 0 \text{ along all connections}$$

A_3 and B_3^- cycles

A_3 and B_3^- cycles are geometrically identical.



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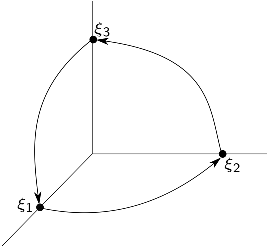
Symmetries $\kappa_i, \kappa_{ij}, \kappa_{ijk} : \mathbb{R}^4 \to \mathbb{R}^4$

$$\kappa_1(x_1, x_2, x_3, x_4) = (x_1, -x_2, -x_3, -x_4)$$



	A_3 cycle	B_3^- cycle
symmetry Γ	$\langle \kappa_{12}, \kappa_{23}, \kappa_{34} \rangle \cong \mathbb{Z}_2^3$	$\langle \kappa_1, \kappa_2, \kappa_3, \kappa_4 \rangle \cong \mathbb{Z}_2^4$
fixed-point spaces	lines ($\mathbb{Z}_2^2$), planes ($\mathbb{Z}_2$)	lines ($\mathbb{Z}_2^3$), planes ($\mathbb{Z}_2^2$), spheres ($\mathbb{Z}_2$)
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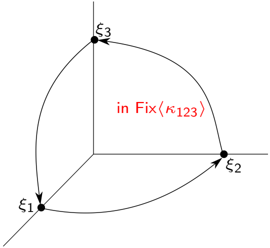

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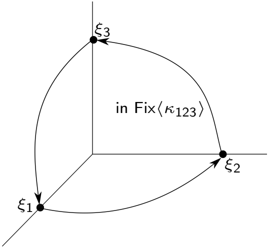

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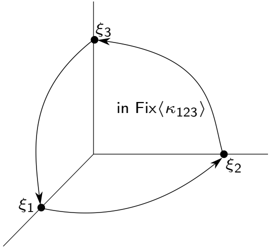

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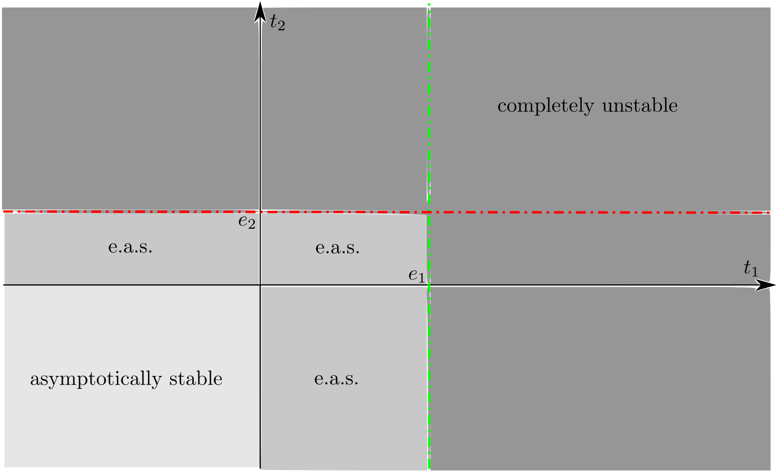
$\rightarrow$ How much do the stability properties of these cycles differ?

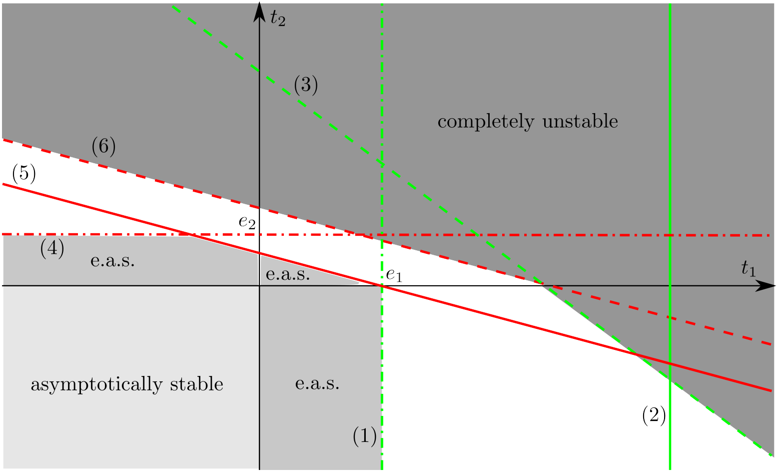
$$\dot{x}_j = a_j x_j + \left(\sum_{i=1}^{4} b_{1i} x_i^2\right) x_j + c_j x_1 x_2 x_3 x_4 x_j$$

A_3 and B_3^- cycles – stability



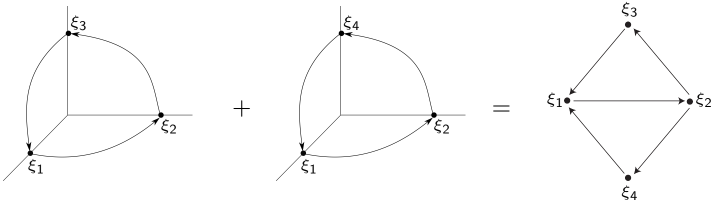
A_3-cycles, $t_3 < 0$

A_3 and B_3^- cycles – stability



B_3^--cycles, $t_3 < -c_3 < 0$

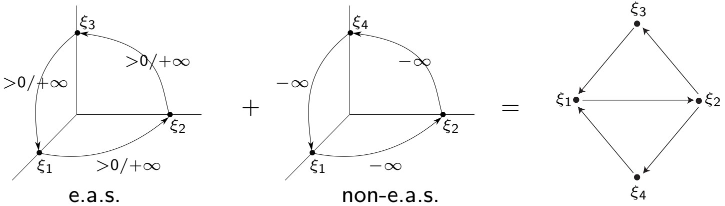
(A_3, A_3) and (B_3^-, B_3^-) networks



The corresponding networks are also geometrically identical ...

...but their stability properties differ:

(A_3, A_3)



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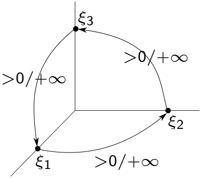
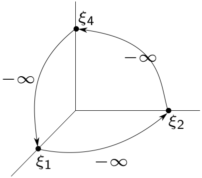


e.a.s.

$+$



non-e.a.s.

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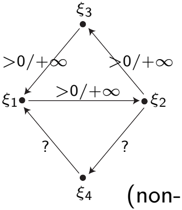

(non-)e.a.s.?

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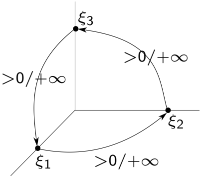

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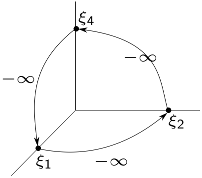

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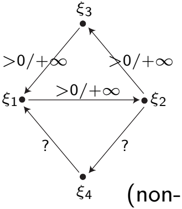
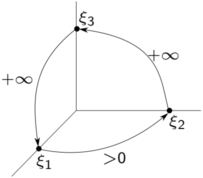


(non-)e.a.s.?

(B_3^-, B_3^-)
(1a)



e.a.s.

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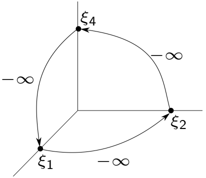

non-e.a.s.

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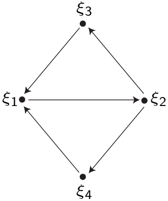
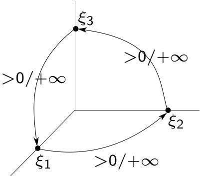


(A_3, A_3) and (B_3^-, B_3^-) networks

…but their stability properties differ:

(A_3, A_3)



e.a.s.

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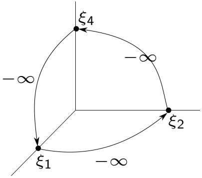

non-e.a.s.

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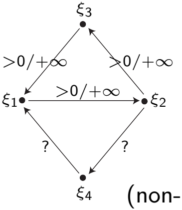

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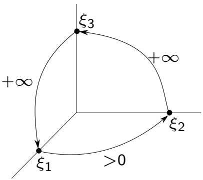
(1a)



e.a.s.

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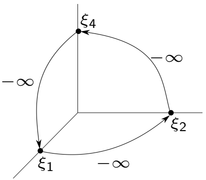

non-e.a.s.

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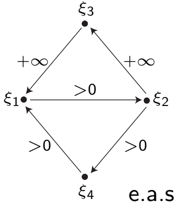

e.a.s.

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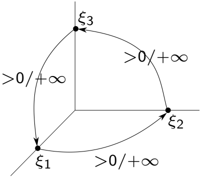
(A_3, A_3)



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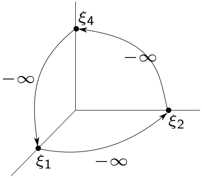

non-e.a.s.

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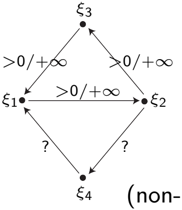
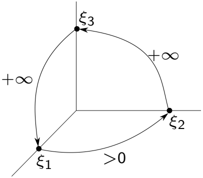


(non-)e.a.s.?

(B_3^-, B_3^-)
(1b)



e.a.s.

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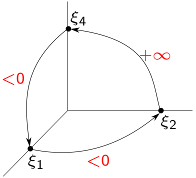

non-e.a.s.

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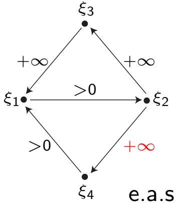

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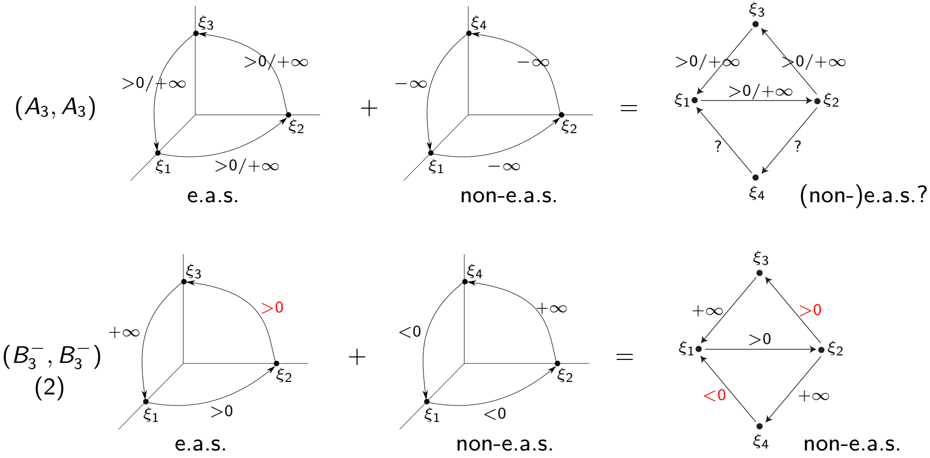

(A_3, A_3) and (B_3^-, B_3^-) networks

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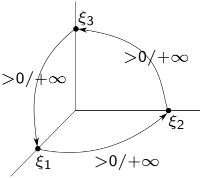
(A_3, A_3)



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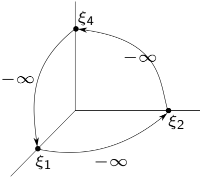

non-e.a.s.

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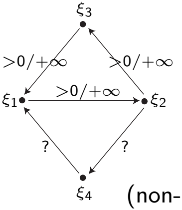
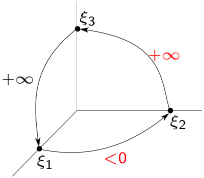


(non-)e.a.s.?

(B_3^-, B_3^-)
(3)



non-e.a.s.

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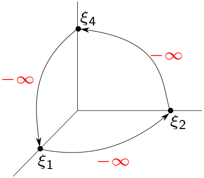

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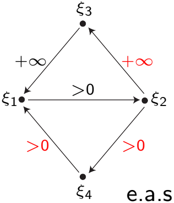

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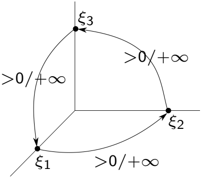

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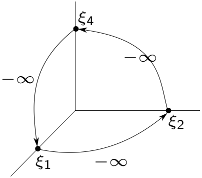

non-e.a.s.

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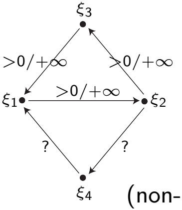
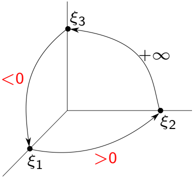


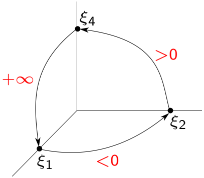
(non-)e.a.s.?

(B_3^-, B_3^-)
(4)



non-e.a.s.

+



non-e.a.s.

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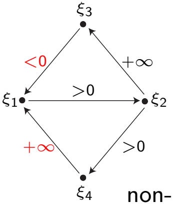
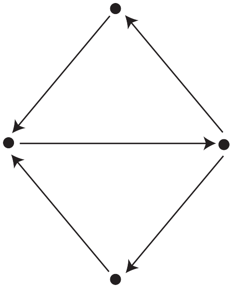


non-e.a.s.

There is no switching near the (B_3^-, B_3^-) network.

→ Is this due to the common connection and/or type B?

→ Is switching possible in the (A_3, A_3) network?

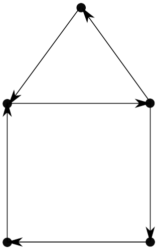


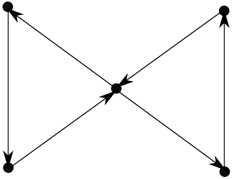
There is no switching near the (B_3^-, B_3^-) network.

→ Is this due to the common connection and/or type B?

→ Is switching possible in the (A_3, A_3) network?

→ Or in these networks that can be realized in $\mathbb{R}^5$?





Summary

- There are eight simple . . . heteroclinic networks in $\mathbb{R}^4$.
- Stability properties of geometrically identical cycles of different types depend on the symmetry group Γ:
  - less symmetry (type A) – uniform stability along connections, all stability indices have the same sign
  - more symmetry (type B) – varying stability configurations, stability indices with different sign possible
- Switching seems to depend on dimension, the presence of a common connection and the type of the network.

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Thank you very much for your attention.