Large deviation estimates for Markovian cocycles

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(Base) Large Deviation Estimates

Consider an ergodic dynamical system $(T, X, \mathcal{F}, \mathbb{P})$.

We say that a measurable observable $\xi : X \to \mathbb{R}$ satisfies **large deviation estimates of exponential type** if there exist constants $C, k, \varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and $n \in \mathbb{N}$,

$$\mathbb{P}\left\{ \,|\, \frac{1}{n} \sum_{j=0}^{n-1} \xi \circ T^j - \int \xi \, d\mathbb{P} \,|\, > \varepsilon \right\} < C\, e^{-k\, \varepsilon^2\, n}$$

(Fiber) Large Deviation Estimates

Consider a linear cocycle $A : X \to \mathrm{Mat}(\mathrm{d}, \mathbb{R})$ over a base ergodic dynamical system $(T, X, \mathcal{F}, \mathbb{P})$.

We say that $A : X \to \mathrm{Mat}(\mathrm{d}, \mathbb{R})$ satisfies **large deviation estimates of exponential type** if there exist constants $C, k, \varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and $n \in \mathbb{N}$,

$$\mathbb{P}\left\{ \, \left| \frac{1}{n} \log \|A^{(n)}\| - L_1(A) \right| > \varepsilon \, \right\} < C \, e^{-k \, \varepsilon^2 \, n}$$

$$A^{(n)}(x) := A(T^{n-1}x) \ldots A(Tx) \, A(x) \, ,$$

$$L_1(A) := \lim_{n \to +\infty} \frac{1}{n} \int \log \|A^{(n)}(x)\| \, d\mathbb{P}(x) \, .$$

The dependence of the constants C, k, ε_0 on A will be crucial.

Large Deviation Principle

Consider an ergodic dynamical system $(T, X, \mathcal{F}, \mathbb{P})$.

We say that a measurable observable $\xi : X \to \mathbb{R}$ satisfies a **large deviation principle** if there exist a function $I(\varepsilon) > 0$, for $\varepsilon > 0$, called the **rate function**, such that

$$\lim_{n\to+\infty} \frac{1}{n} \log \mathbb{P} \left\{ \left| \frac{1}{n} \sum_{j=0}^{n-1} \xi \circ T^j - \int \xi \, d\mathbb{P} \right| > \varepsilon \right\} = -I(\varepsilon) \, .$$

Continuity of Lyapunov Exponents

Let $\mathcal{C}$ be a space of linear cocycles over some ergodic dynamical system $(T, X, \mathcal{F}, \mathbb{P})$.

Theorem (with S. Klein)

The continuity of the Lyapunov exponents and that of the Oseledets decomposition follows from:

1. *base large deviation type estimates for* $(T, X, \mathcal{F}, \mathbb{P})$,
2. **uniform fiber large deviation type estimates** *for* A.

When Lyapunov exponents are simple an explicit modulus of continuity holds which depends on the type of large deviations.

First Application

Theorem (with Silvius Klein)

Given $\omega \in \mathrm{DC}(\mathbb{T}^n)$*, the Lyapunov exponents and the Oseledets decomposition of a quasi-periodic cocycle* $A : \mathbb{T}^n \to \mathrm{Mat}(d, \mathbb{R})$*, with* $\det A \not\equiv 0$*, over the torus translation* $T : \mathbb{T}^n \to \mathbb{T}^n$*,* $T(x) = x + \omega$*, are continuous functions of* $A \in C^\omega(\mathbb{T}^n, \mathrm{Mat}(\mathrm{d}, \mathbb{R}))$*.*

When the Lyapunov exponents are simple some explicit modulus of weak-Hölder (Hölder for $n = 1$*) continuity holds.*

Quasi-periodic Cocycles (Continuity Theorems)

Analytic Schrödinger cocycles ($d = 2$)

1. M. Goldstein, W. Schlag (2001) (Hölder continuity in E, Diophantine frequency ω, $\mathbb{T}^1$)
2. J. Bourgain, S. Jitomirskaya (2002) (joint continuity in (ω, E), arbitrary frequency $\omega \notin \mathbb{Q}$, $\mathbb{T}^1$)
3. J. Bourgain (2005) (joint continuity in (ω, E), arbitrary frequency $\omega \notin \mathbb{Q}$, $\mathbb{T}^n$)

Analytic GL-cocycles

4. S. Jitomirskaya, C. A. Marx (2012) (joint continuity, arbitrary frequency $\omega \notin \mathbb{Q}$, $\mathrm{Mat}(2)$-cocycles with singularities, $\mathbb{T}^1$)
5. A. Ávila, S. Jitomirskaya, C. Sadel (2014) (joint continuity, arbitrary frequency $\omega \notin \mathbb{Q}$, $\mathbb{T}^1$, $\mathrm{Mat(d)}$-cocycles with singularities)
6. P. Duarte, S. Klein (2014) (quantitative continuity, Diophantine frequency ω, $\mathbb{T}^n$, $\mathrm{GL}(d)$-cocycles)

Second Application

Let $P \in \mathrm{Mat}(\mathrm{n}, \mathbb{R})$ be an **irreducible aperiodic stochastic** matrix, and consider the associated **mixing Markov shift** $(T : X \to X, \mathbb{P})$, where $X = \{1, \ldots, n\}^{\mathbb{Z}}$.

Consider the space $\mathcal{C}$ of all measurable bounded cocycles $A : X \to \mathrm{GL}(d, \mathbb{R})$ with $A(x) = A(x_0, x_1)$.

Theorem (with Silvius Klein)

On the subspace of **irreducible** *cocycles the Lyapunov exponents and the Oseledets decomposition are continuous functions of the cocycle $A \in \mathcal{C}$.*

When the Lyapunov exponents are simple an explicit Hölder modulus of continuity holds.

Random (I.I.D.) Cocycles (Continuity Theorems)

Generic cocycles

1. H. Furstenberg and Y. Kifer (1983) (generic cocycles)
2. E. Le Page (1989) (irreducible contracting cocycles)

General cocycles

3. Y. Peres (1991) (analyticity in the probabilities)
4. C. Bocker-Neto and M. Viana (2010) (joint continuity, $\mathrm{GL}(2)$-cocycles)
5. E. Malheiro, M. Viana (2014) (joint continuity, locally constant $\mathrm{GL}(2)$-cocycles over a Markov shift)
6. A. Ávila, A. Eskin, M. Viana ($2015 \pm \varepsilon$) (joint continuity, locally constant $\mathrm{GL}(d)$-cocycles over a Bernoulli shift)

General Processes

1. H. Cramér (1938) (iid Large deviation principle)
2. S. V. Nagaev (1957) (Markov chains)

Irreducible Contracting Cocycles

3. V. N. Tutubalin (1965) (random iid cocycles)
4. E. Le Page (1981) (random iid cocycles)
5. P. Bougerol (1988) (cocycles over Markov systems)

General Processes

6. H. Henion, L. Hervé (2001) (abstract setting to prove limit theorems by spectral method)

Markov Shifts

Σ compact metric space
$K : \Sigma \to \mathrm{Prob}(\Sigma)$ stochastic kernel with stationary measure μ

$X = \Sigma^{\mathbb{Z}}$ space of sequences
$T : X \to X$ shift map
The random variable $e : X \to \Sigma$, $e(x) := x_0$, and the shift T generate the random process $\{e \circ T^n\}_{n \geq 0}$

$\mathbb{P}$ is the unique probability in X such that $\{e \circ T^n\}_{n \geq 0}$ is a stationary Markov process with common distribution μ and transition kernel K.

The dynamical system $(T : X \to X, \mathbb{P})$ is called a **Markov shift**.

The sum processes

Given $\xi : \Sigma \to \mathbb{R}$ measurable, define
$\hat{\xi} : X \to \mathbb{R}, \quad \hat{\xi}(x) := \xi(x_0)$.

The random variable ξ determines the following **sum process**

$$S_n(\xi) := \sum_{j=0}^{n-1} \hat{\xi} \circ T^j$$

Notice that

$$\frac{1}{n}\, \mathbb{E}[S_n(\xi)] = \int \xi \, d\mu =: \mathbb{E}_\mu(\xi) \, .$$

Cramér's Argument

By Chebyshev's inequality

$$\mathbb{P}\{\, S_n(\xi) > n\,\varepsilon \,\} \leq e^{-t\,\varepsilon\,n}\, \mathbb{E}[e^{t\,S_n(\xi)}]$$

$\eta_n(t) := \mathbb{E}[e^{t\,S_n(\xi)}]$ is the **moment generating function** of $S_n(\xi)$.

In the iid case,

$$\mathbb{E}[e^{t\,S_n(\xi)}] = \mathbb{E}[e^{t\,\hat{\xi}}]^n = e^{n\,c(t)}$$

where $c(t) = \log \mathbb{E}[e^{t\,\hat{\xi}}]$ is the **second moment generating function** of $\hat{\xi}$, a convex function such that

$c(0) = 0$ and $c'(0) = \mathbb{E}_\mu(\xi)$.

Cramér's Argument

Hence, assuming $\mathbb{E}_\mu(\xi) = 0$, the convex function $c(t)$ attains its absolute minimum at $t = 0$ and

$$\mathbb{P}\{\, S_n(\xi) > n\,\varepsilon \,\} \leq e^{-t\,\varepsilon\,n}\, e^{n\,c(t)} = e^{-n\,(t\,\varepsilon - c(t))}$$

The **Legendre transform** of the convex function $c(t)$ is

$$\hat{c}(\varepsilon) := \max_t \left(\varepsilon\, t - c(t)\right) .$$

Choosing the value of t that maximizes $\varepsilon\, t - c(t)$ we get

$$\mathbb{P}\{\, S_n(\xi) > n\,\varepsilon \,\} \leq e^{-\hat{c}(\varepsilon)\, n}$$

Given an observable $\xi : \Sigma \to \mathbb{R}$, the operators in the family

$$(Q_t f)(x) := \int_\Sigma f(y)\, e^{t\xi(y)}\, K(x, dy)$$

are called **Laplace-Markov operators**.

For $t = 0$,

$$(Q_0 f)(x) := \int_\Sigma f(y)\, K(x, dy)$$

is a **Markov operator**.

The moment generating function of $S_n(\xi)$ can be expressed as

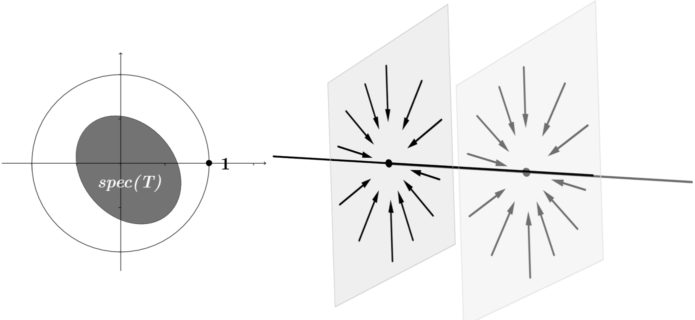
$$\mathbb{E}[e^{t\, S_n(\xi)}] = \mathbb{E}[Q_t^n \mathbf{1}]$$

Quasi-Compactness

The Markov operator $Q_0 : \mathcal{B} \to \mathcal{B}$ satisfies:

- $Q_0 \mathbf{1} = \mathbf{1}$,
- Q_0 is positive, $\quad f \geq 0 \quad \Rightarrow \quad Q_0 f \geq 0$
- $\mathrm{spec}(Q_0) \subset \mathbb{D}_1(0)$

We assume $Q_0 : \mathcal{B} \to \mathcal{B}$ is **quasi-compact** and **simple**, i.e., there exists a Q_0-invariant decomposition $\mathcal{B} = \langle \mathbf{1} \rangle \oplus \mathcal{H}$ such that $Q_0|_{\mathcal{H}}$ has spectral radius < 1.



Quasi-Compactness

Q_0 quasi-compact simple $\Rightarrow$ Q_t quasi-compact simple, for small t
Q_t is positive for t real
Hence there exists an eigenvalue $\lambda(t) > 0$,
and an eigen-function $v(t) \in \mathcal{B}$ such that

$$Q_t v(t) = \lambda(t)\, v(t) \quad \text{and} \quad \mathbb{E}(v(t)) = 1$$

Therefore,

$$\mathbb{E}[e^{t\, S_n(\xi)}] = \mathbb{E}[Q_t^n \mathbf{1}] \approx \mathbb{E}[Q_t^n v(t)] = \lambda(t)^n\, \mathbb{E}[v(t)] = e^{n\, c(t)} \,,$$

where $c(t) = \log \lambda(t)$ is a convex function such that

$$c(0) = 0 \text{ and } c'(0) = \mathbb{E}_\mu(\xi).$$

References

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