Bifurcations from homoclinic cycles

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Consider a differential equation

$$\dot{\mathbf{x}} = f(\mathbf{x})$$

on a Euclidean space $\mathbb{R}^n$,

that is equivariant

under the representation of a discrete group G, i.e.

$$gf(\mathbf{x}) = f(g\mathbf{x}), \qquad \forall g \in G.$$

Homoclinic cycles

Heteroclinic networks

A heteroclinic network is a connected set consisting of finitely many disjunct equilibria p_j, $j = 1, \ldots, \ell$, and heteroclinic trajectories $\gamma_j(t)$, $j = 1, \ldots, k$ between them.

Homoclinic cycles

A *homoclinic cycle* Γ is a connected invariant set that is equal to a group orbit $\langle h \rangle \bar{\gamma}$, for a heteroclinic trajectory γ connecting p to hp for some $h \in G$.

Codimension

The codimension of a homoclinic cycle is the number p of parameters to find it in generic p-parameter families.

Homoclinic cycles

The connectivity matrix

$C = (c_{ij})$ of a heteroclinic network with heteroclinic trajectories γ_i, $i = 1, \ldots, k$, is a 0-1 matrix, that is $c_{ij} \in \{0, 1\}$, where $c_{ij} = 1$ if and only if the endpoint (the ω-limit $\omega(\gamma_i)$) of the heteroclinic connection γ_i is equal to the starting point (the α-limit $\alpha(\gamma_j)$) of the heteroclinic connection γ_j.

Topological Markov chains

Let $\Sigma_k = \{1, \ldots, k\}^{\mathbb{Z}}$ denote the set of double infinite sequences $\kappa : \mathbb{Z} \to \{1, \ldots, k\}$, $i \mapsto \kappa_i$.
Let $C = (c_{ij})_{i,j \in \{1,\ldots,k\}}$ be a 0-1 matrix.

$$\Sigma_C = \{\kappa \in \Sigma_k \mid a_{\kappa_i \kappa_{i+1}} = 1\},$$

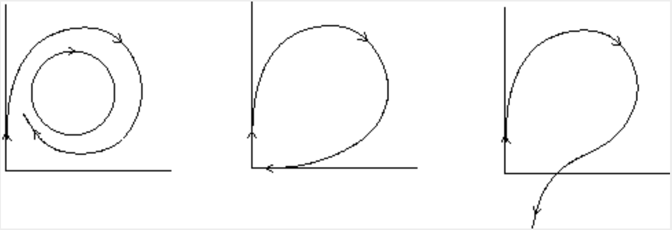
With time in $\mathbb{N}$: Σ_k^+ and Σ_C^+.

Take a heteroclinic network with heteroclinic trajectory γ_i and a open neighborhood $\mathcal{U}$ of it.
Take cross sections S_i transverse to γ_i and write Π for the first return map on the collection of cross sections $S := \cup_{j=1}^{k} S_j$.

Realizations

Let κ be a symbolic sequence in Σ_C. We call a trajectory $\mathcal{O}$ a *realization of κ in $\mathcal{U}$*, if $\mathcal{O} \subset \mathcal{U}$ and if there is an $x_\kappa \in \mathcal{O}$ such that $\Pi^i(x_\kappa) \in S_{\kappa_i}$, $i \in \mathbb{Z}$.

Goal: derive symmetric equivalent of codimension one homoclinic bifurcation



References

A.J. Homburg, A.C. Jukes, J. Knobloch, J.S.W. Lamb
Bifurcation from codimension one relative homoclinic cycles
Transactions of the Americal Mathematical Society (2011)

Codimension one connections

Let $\dot{x} = f(x, \lambda)$ be a one parameter family of differential equations equivariant with respect to a finite group G, with the following properties:

1. At $\lambda = 0$, there is a codimension one homoclinic cycle Γ with hyperbolic equilibria.
2. The connecting trajectories in Γ are nondegenerate.
3. The isotropy subgroup G_p of an equilibrium p in Γ acts absolutely irreducibly on the leading stable eigenspace at p, and the real leading stable eigenvalues of the linearized vector field about p are closest of all eigenvalues to the imaginary axis.
4. The connecting trajectories $\gamma_1, \ldots, \gamma_k$ in Γ approach the equilibria along the leading stable directions (a non-orbit-flip condition). The connecting trajectories in Γ satisfy a non-inclination-flip condition.

Codimension one connections

There is an explicit construction of $k \times k$ matrices A_- and A_+ with coefficients in $\{0, 1\}$ and the nonzero coefficients in mutually disjoint positions, so that the following holds for any generic family unfolding the homoclinic cycle.

Take cross sections S_i transverse to γ_i and write Π_λ for the first return map on the collection of cross sections $\cup_{j=1}^k S_j$.

For $\lambda > 0$ small, there is an invariant set $\mathcal{D}_\lambda \subset \cup_{j=1}^k S_j$ for Π_λ such that for each $\kappa \in \Sigma_{A_+}$ there exists a unique $x \in D_\lambda$ with $\Pi_\lambda^i(x) \in S_{\kappa_i}$.
$(\mathcal{D}_\lambda, \Pi_\lambda)$ is topologically conjugate to (Σ_{A_+}, σ).

Analogous statements hold for $\lambda < 0$, Σ_{A_+} replaced by Σ_{A_-}.

Consider a heteroclinic chain γ_{κ_i} with $p_{k_i} = \omega(\gamma_{\kappa_{i-1}}) = \alpha(\gamma_{\kappa_i})$
$\alpha(\gamma_i), \omega(\gamma_i)$ stand for begin and end point of the connection γ_i

What goes into the proof: Lin's method

For ω_i large, there is a unique piecewise continuous trajectory, with jumps in S_{κ_i} in specified direction (perpendicular to stable and unstable manifolds) with transition time ω_i between sections $S_{\kappa_{i-1}}$ and S_{κ_i}
Bifurcation equation for trajectories: "jumps = zero"

By the conditions there are well defined solutions
η_i^s : a bounded solution of the variational equation along γ_i.
η_j^l : a bounded solution of the adjoint variational equation along γ_j.

Limits of these bounded solutions with time going to $\pm\infty$ yield vectors e_i^s, e_j^- in the leading stable spaces.

After reparameterization, the bifurcation equations have a form

$$\lambda - e^{2\mu^s \omega_i} \langle e^s_{\kappa_{i-1}}, e^-_{\kappa_i} \rangle + h.o.t. = 0$$

These can be solved if the inner products do not vanish

Definition A^+, A^-

$M = (m_{ij}), 1 \leq i, j \leq k$, with

$$m_{ij} = \begin{cases} 0, & \omega(\gamma_i) \neq \alpha(\gamma_j), \\ \mathrm{sgn}\ \langle e_i^s, e_j^- \rangle, & \omega(\gamma_i) = \alpha(\gamma_j). \end{cases}$$

$A^+ = \frac{1}{2}(M + |M|)$, $A^- = -\frac{1}{2}(M - |M|)$

This description of the dynamics provides a complete picture of the local nonwandering dynamics near Γ if and only if

$$A_+ + A_- = C. \tag{1}$$

Figure: Illustration of three homoclinic trajectories approaching p when $\dim(E^s_p) = 2$. The state space has to be at least four dimensional.

Example with $\mathbb{D}_3$ symmetry

Consider a homoclinic cycle with three connecting trajectories γ_i, $i = 1, 2, 3$, to a hyperbolic equilibrium, such that each has isotropy equal to $\mathbb{Z}_2$.

As all three connecting trajectories are connecting to the same equilibrium, the connectivity matrix C is given by

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Suppose that the absolutely irreducible action of the group $\mathbb{D}_3$ on the leading eigenspace is

1. one-dimensional and trivial, or,
2. two-dimensional and acting as the symmetry group of the equilateral triangle.

Codimension one connections

In case the representation of $\mathbb{D}_3$ on the leading eigenspace is trivial, the leading eigenspace will generically be one-dimensional so that the connecting trajectories γ_i $(i = 1, 2, 3)$ come into the equilibrium in the same direction (tangent to each other). The matrices A_-, A_+ are given by

$$A_- = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad A_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so that there is a nonwandering set that is topologically conjugate to the suspension of a full shift on three symbols if $\lambda < 0$ and no nontrivial nonwandering dynamics near the relative homoclinic cycle if $\lambda > 0$.

Codimension one connections

In case the representation of $\mathbb{D}_3$ on the leading eigenspace is nontrivial, the leading eigenspace will generically be two-dimensional with the connecting trajectories γ_i, $i = 1, 2, 3$, coming into the equilibrium in three different directions (each separated by an angle of $2\pi/3$).
The matrices A_-, A_+ are given by

$$A_- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A_+ = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

If $\lambda < 0$ the nonwandering set consists of three periodic solutions, shadowing the individual connecting trajectories γ_i $(i = 1, 2, 3)$. If $\lambda > 0$ the nonwandering dynamics is more complicated, with all trajectories avoiding shadowing twice the same connecting trajectory.

References

A.J. Homburg, M. Kellner, J. Knobloch
Construction of codimension one homoclinic cycles
Dynamical Systems (2014)

If $A_+ + A_- \neq C$ then the the theorem may not fully describe the bifurcation and dynamics
This can be forced by symmetry: e.g. with $\mathbb{D}_4$ symmetry inner products $\langle e_i^s, e_j^- \rangle$ may be forced to be zero.

There is an explicit construction of polynomial vector fields in $\mathbb{R}^4$ with $\mathbb{D}_m$ symmetry and m homoclinic solutions to the origin.

Codimension one connections

Polynomial vector fields on $\mathbb{R}^4$ with $\mathbb{D}_4$ symmetry and 4 homoclinic trajectories to the origin, plus unfolding in λ

$$f_4(x,\lambda) =$$

$$\begin{pmatrix} ax_1 & +bx_2 & -ax_1^3 + 3ax_1y_1^2 & \\ ay_1 & +by_2 & +3ax_1^2y_1 - ay_1^3 & \\ bx_1 & +ax_2 & -2bx_1^3 + 6bx_1y_1^2 & -2ax_1^2x_2 + 2ay_1^2x_2 + 4ax_1y_1y_2 \\ by_1 & +ay_2 & +6bx_1^2y_1 - 2by_1^3 & +4ax_1y_1x_2 + 2ay_2(x_1^2 - y_1^2) \end{pmatrix}$$

$$+\lambda \begin{pmatrix} 2x_1 & -4x_1^3 + 12x_1y_1^2 \\ 2y_1 & +12x_1^2y_1 - 4y_1^3 \\ -2x_2 & \\ -2y_2 & \end{pmatrix}.$$

References

P.J. Holmes
A strange family of three-dimensional vector fields near a degenerate singularity
J. Differential Equations (1980)

A.J. Homburg, J. Knobloch
Switching homoclinic networks
Dynamical Systems (2010)

Connectivity matrix

A heteroclinic network Γ with connectivity matrix C is switching (forward switching) if for each sequence $\kappa \in \Sigma_C$ ($\kappa \in \Sigma_C^+$) and each tubular neighborhood $\mathcal{U}$ of Γ, there exists a realization of κ in $\mathcal{U}$.

Define the representation of the group $G = \mathbb{Z}_2 \ltimes \mathbb{Z}_2^2$ on $\mathbb{R}^5$ as follows: $\mathbb{Z}_2 = \mathbb{Z}_2(g_0)$ is acting by

$$g_0(x, y_1, y_2, z_1, z_2) = (-x, z_1, z_2, y_1, y_2)$$

and $\mathbb{Z}_2^2 = \mathbb{Z}_2^2(g_1, g_2)$ is acting by

$$g_1(x, y_1, y_2, z_1, z_2) = (x, -y_1, -y_2, z_1, z_2),$$
$$g_2(x, y_1, y_2, z_1, z_2) = (x, y_1, y_2, -z_1, -z_2).$$

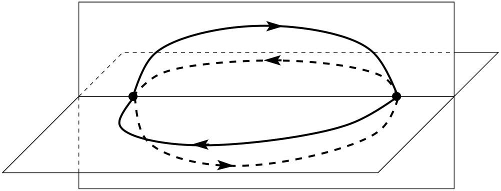
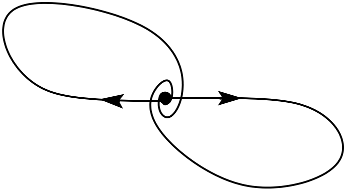


Figure: Homoclinic network Γ (above) and reduced homoclinic network (identifying equilibria; below)



The connectivity matrix C of Γ reads

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Theorem

Consider an ordinary differential equation on $\mathbb{R}^5$ that is G-equivariant and possesses a homoclinic network Γ as above. Assume eigenvalue conditions $\lambda^d, \lambda^r < Re\, \lambda^c < -\lambda^e$. Then Γ is a robust asymptotically stable homoclinic cycle which is forward switching.

Open problem: what realizations occur for typical initial points, with respect to Lebesgue measure?

Theorem

Consider an ordinary differential equation on $\mathbb{R}^5$ that is G-equivariant and possesses a homoclinic network Γ as above. Assume $\lambda^d, \lambda^r < -\lambda^e < Re\,\lambda^c$. Then Γ is a robust homoclinic cycle which is switching.

Essential asymptotic stability

References

R. Driesse, A.J. Homburg
Essentially asymptotically stable homoclinic networks
Dynamical Systems (2009)

Essentially asymptotically stable invariant sets

A flow-invariant compact set H is essentially asymptotically stable if for any open neighborhood U of H there is a set C so that for any given number $a \in (0,1)$ there is an open ε-neighborhood $V \subset U$ of H such that all trajectories starting in $V - C$ remain in U and are asymptotic to H and $\mu(V - C)/\mu(V) > a$, where μ is the Lebesgue measure.

Equivariance on $\mathbb{R}^5$

Consider differential equations on $\mathbb{R}^5$, equivariant under $G \cong \mathbb{Z}_2^3 \rtimes \mathbb{Z}_4$ generated by

$$
\begin{aligned}
g_1 &: (x, y, z, u, v) \to (x, y, -z, u, v), \\
g_2 &: (x, y, z, u, v) \to (x, -y, z, u, v), \\
g_3 &: (x, y, z, u, v) \to (x, y, z, u, -v), \\
h &: (x, y, z, u, v) \to (-x, z, y, -v, u).
\end{aligned}
$$

Essential asymptotic stability

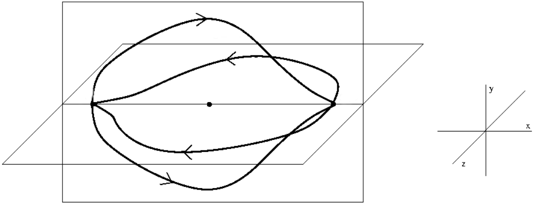


Figure: Homoclinic network Γ inside three dimensional fixed point space $R = \{u, v = 0\}$

Essential asymptotic stability

Theorem

Let $\dot{\mathbf{x}} = f(\mathbf{x})$ be a G-equivariant differential equation on $\mathbb{R}^5$, $G \cong \mathbb{Z}_2^3 \rtimes \mathbb{Z}_4$ generated by g_1, g_2, g_3 and h, possessing a robust homoclinic network $G\Gamma$ contained in the fixed point space R as above. Assume eigenvalue conditions $0 < \lambda_v/\lambda_y < \min\{\lambda_u/\lambda_z, 1\}$, $\lambda_u < 0$ and $-\lambda_z/\lambda_y > 1$. Then $G\Gamma$ is essentially asymptotically stable.

An explicit example is contained in

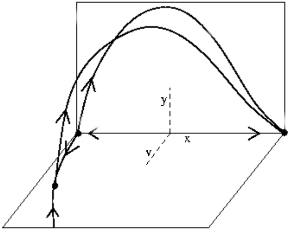
the family of differential equations

$$\begin{aligned}
\dot{x} &= \nu x + z^2 - y^2 - x^3 + \beta x(y^2 + z^2), \\
\dot{y} &= y(\lambda + ay^2 + bz^2 + cx^2) + yx, \\
\dot{z} &= z(\lambda + az^2 + by^2 + cx^2) - zx, \\
\dot{u} &= \mu u + dux, \\
\dot{v} &= \mu v - dvx.
\end{aligned}$$

Essential asymptotic stability

Transverse bifurcation from asymptotically stable to essentially asymtotically stable homoclinic cycle

Let $\dot{\mathbf{x}} = f(\mathbf{x}, \varepsilon)$ be a G-equivariant system of differential equations depending on a parameter ε. Assume that it possesses a robust homoclinic network $G\Gamma$ as above, asymptotically stable for $\varepsilon < 0$, with at $\varepsilon = 0$ a supercritical transverse bifurcation; $\lambda_v(0) = 0$, $-\lambda_z(0) > \lambda_y(0) > 0$, $\lambda_x(0) < 0$, $\lambda_u(0) < 0$.
Assume generic unfolding conditions.



Then for $\varepsilon > 0$ small enough the heteroclinic network $\mathcal{A} = G(W^u(p') \cup W^u(p))$ is an attractor. Furthermore there is an open invariant neighborhood U of $G\Gamma$, so that for $\varepsilon > 0$ small enough, trajectories starting at $\mathbf{x} \in U - GW^s(p')$ converge for positive time to $G\Gamma$.