The structure of balanced big CM modules over CM rings

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SS 32: *Homological and Combinatorial Commutative Algebra*

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Example (Lazard, 1969)

Let $R = k[\![x, y, z]\!]/(xz, yz, z^2)$. There is an R-module M with $\operatorname{fd}_R M = 1$ such that M can not be written as a direct limit $M = \varinjlim M_i$ of finitely generated modules M_i with $\operatorname{fd}_R M_i \leqslant 1$.

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small CM $=$ finitely generated (balanced) big CM – or zero.

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Question

Do all big CM modules share some kind of common structure?

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Every balanced big CM module can be obtained as a direct limit of small CM modules.

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For every R-module M, the following conditions are equivalent:

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Theorem A is a consequence of:

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For every R-module M, the following conditions are equivalent:

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(ii) Every s.o.p. for R is a weak M-regular sequence.

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Enochs, Jenda, and Torrecillas defined Gorenstein flat modules.

Application I (regular rings) 1/3

Recall that an R-module M is called balanced big CM if

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An R-module M is called weak balanced big CM if

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Now the equivalence (i)$\Leftrightarrow$(ii) in Theorem B can be phrased as:

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translates into:

A classic text book theorem

Over a PID one has:

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Application I (regular rings) 3/3

Corollary of Theorem B

The following conditions are equivalent:

(i) R is regular.

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(ii) $\Rightarrow$ (i): By (ii) every small CM R-module is flat and hence projective (as it is finitely generated). Thus R is regular. □

Application II (covers) 1/3

Theorem (Auslander and Buchweitz, 1989)

Every finitely generated R-module M has a maximal CM approximation, that is, there exists a short exact sequence,

$$0 \longrightarrow I \longrightarrow X \stackrel{\pi}{\longrightarrow} M \longrightarrow 0,$$

where X is small CM and I has finite injective dimension.

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$$\begin{array}{ccc} & & X' \\ & \overset{\exists\,\alpha}{\swarrow} & \Big\downarrow \forall\,\pi' \\ X & \xrightarrow[\pi]{} & M \end{array}$$

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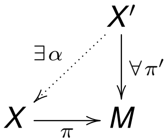

Every **finitely generated** R-module has a surjective precover w.r.t. the class $\mathcal{C} = \{\text{small CM modules}\}$.

Application II (covers) 2/3

Combined with a result of Yoshino/Takahashi, this implies:

If R is complete, then every **finitely generated** R-module has a surjective cover w.r.t. $\mathcal{C} = \{\text{small CM modules}\}$.

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Here is another result in the same direction:

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Application II (covers) 3/3

Remark.

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Every R-module M with $\mathfrak{m}M \neq M$ has a surjective cover w.r.t.

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Application III (CM preenvelopes) 1/4

Theorem B implies the existence of CM preenvelopes:

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which is easily seen to be closed under products. □

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commutes, we can use $\alpha = \delta_X^{-1}\varepsilon^{\dagger\dagger}$. □

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$$k^{\dagger\dagger} \cong \begin{cases} k & \text{if} \quad d = 0 \\ 0 & \text{if} \quad d > 0 \end{cases} \qquad \text{is CM.}$$

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$$k^{\dagger\dagger} \cong \begin{cases} k & \text{if} \quad d = 0 \\ 0 & \text{if} \quad d > 0 \end{cases} \qquad \text{is CM.}$$

Thus a CM preenvelope of k is:

$$\begin{cases} k \xrightarrow{=} k & \text{if} \quad d = 0 \\ k \to 0 & \text{if} \quad d > 0 \quad \text{(not injective)} \end{cases}$$

Application III (CM preenvelopes) 4/4

The case $M = \mathfrak{m}$

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Application III (CM preenvelopes) 4/4

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Application III (CM preenvelopes) 4/4

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Thus a CM preenvelope of $\mathfrak{m}$ is:

$$\begin{cases} \mathfrak{m} \xrightarrow{=} \mathfrak{m} & \text{if} \quad d = 0, 1 \\ \mathfrak{m} \hookrightarrow R & \text{if} \quad d > 1 \end{cases}$$

More about CM preenvelopes 1/3

A well-known fact:

High syzygies are CM

Set $d = \dim R$ and let M be a finitely generated R-module. If

$$\cdots \to X_1 \to X_0 \to M \to 0$$

is an exact sequence where each X_i is CM (e.g. free), then

$$K_n = \operatorname{Ker}(X_{n-1} \to X_{n-2}) \text{ is CM} \quad \text{for all} \quad n \geqslant d.$$

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There is a “dual” of this result:

Of course, one can not always construct an exact sequence

$$0 \to M \to X^0 \to X^1 \to \cdots \qquad \text{where } X_i \text{ is CM.}$$

But there is a canonical way to construct such a complex:

More about CM preenvelopes 2/3

- Take an CM preenvelope $M \stackrel{\mu^0}{\to} X^0 \to C^1 \to 0$.

More about CM preenvelopes 2/3

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More about CM preenvelopes 2/3

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More about CM preenvelopes 2/3

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More about CM preenvelopes 2/3

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This gives a (non-exact) complex:

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Theorem (not related to Theorem B)

The modules $C^d, C^{d+1}, C^{d+2}, \ldots$ are all CM.

More about CM preenvelopes 3/3

A CM preenvelope $\mu\colon M \to X$ has the unique lifting property if

$$\begin{array}{ccc} M & \xrightarrow{\mu} & X \\ {\scriptstyle \forall \mu'}\downarrow & \swarrow {\scriptstyle \exists!\, \alpha} & \\ X' & & \end{array} \qquad \text{(unique)}$$

(In this case, μ is a CM envelope.)

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More about CM preenvelopes 3/3

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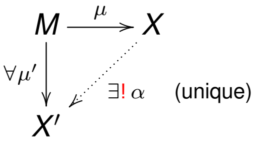

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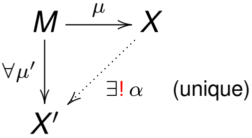

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(iv) $\dim R \leqslant 2$.

On the proof of Theorem B 1/2

Theorem B

For every R-module M, the following conditions are equivalent:

(i) M is a direct limit of small CM R-modules.

(ii) Every s.o.p. for R is a weak M-regular sequence.

(iii) M is Gorenstein flat viewed as a module over $R \ltimes \Omega$.

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(iii) $\Rightarrow$ (i): Follows from work of Enochs and Jenda.

On the proof of Theorem B 2/2

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- As $T = R \ltimes \Omega$ is Gorenstein, one has

$$\operatorname{Gfd}_T M = \sup\{\operatorname{depth} T_{\mathfrak{q}} - \operatorname{depth}_{T_{\mathfrak{q}}} M_{\mathfrak{q}} \mid \mathfrak{q} \in \operatorname{Spec} T\} < \infty.$$

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On the proof of Theorem B 2/2

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for every prime $\mathfrak{p} \subset R$. Hence $\operatorname{Gfd}_{R \ltimes \Omega} M \leqslant 0$. □

Thanks for your attention!