Howson's property for semidirect products of semilattices by groups

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2 ::: Howson group

Howson's Theorem (1954). F free group, H_1, H_2 finitely generated subgroups of F

$$\Longrightarrow \; H_1 \cap H_2 \text{ finitely generated.}$$

Not true in general (**Modalvanksii**):
$G = \langle a, b, c \mid ac = ca, bc = cb \rangle$, $H_1 = \langle a, b \rangle$ and $H_2 = \langle a, bc \rangle$.

G **Howson group** if $H_1 \cap H_2$ is finitely generated whenever $H_1, H_2 \leq_{\text{f.g.}} G$.

3 ::: Howson inverse semigroup

S **Howson inverse semigroup** if $T_1 \cap T_2$ is finitely generated whenever $H_1, H_2 \leq_{\text{f.g.i.}} S$.

S group, Howson group $\iff$ S Howson inverse semigroup.

S Howson *inverse* semigroup $\not\Longrightarrow$ S Howson semigroup.

Jones and Trotter (1989).

- FIS_1 is a Howson inverse semigroup;
- For $n \geq 2$, FIS_n is not a Howson inverse semigroup.

4 ::: Howson inverse semigroup, cont.

Silva (1991).

- $M_1 \cap M_2$ f.g. whichever the monogenic inverse submonoids M_1 and M_2 of any free inverse semigroup.

Margolis and Meakin (1993).

- $C_1 \cap C_2$ f.g. whichever the inverse submonoids C_1 and C_2 of any free inverse monoid, both f.g. as *closed* inverse submonoids.

where

$C \leq M$ is **finitely generated as a closed** inverse submonoid if $C = \langle X \rangle^{\omega}$ for some finite set X, with

$$Y^{\omega} = \{m \in M \colon m \geq y \text{ for some } y \in Y\}.$$

5 ::: Semidirect product

- E semilattice;
- G group;
- $\theta\colon G \to \mathrm{Aut}(E)$ homomorphism: left action $\theta(g)(f) = g \cdot f$.

Semidirect product $E *_\theta G$ with

$$(e, g)(f, h) = (e \wedge (g \cdot f), gh)$$

for all $(e, g), (f, h) \in E \times G$.

- $E *_\theta G$ is a E-unitary inverse semigroup;
- every E-unitary inverse semigroup S embeds into some semidirect product $E *_\theta G$ (**O'Carrol**) with every quotient of S being embeddable in some quotient of $E *_\theta G$ (**Silva**).

6 ::: Preliminares

- E has a fixed point for θ:
  $E *_\theta G$ Howson inverse semigroup $\implies$ G Howson group.

$$\forall g\,,\ g \cdot e = e \implies G \approx \{e\} \times G \hookrightarrow E *_\theta G$$

- E finite:
  $E *_\theta G$ Howson inverse semigroup $\iff$ G Howson group.

$a_x \in A$ $(x \in X \subseteq E *_\theta G)$:

$$i \xrightarrow{a_x} (e_x, \theta_{g_x}) \ \text{ and } \ (f, \pi) \xrightarrow{a_x} (f \wedge \pi(e_x), \pi\theta_{g_x})$$

$u \in \langle X_1 \rangle \cap \langle X_2 \rangle$: $u = (e_u, g_u) = (e_u, h)(h^{-1} \cdot e_u, h^{-1} g_u)$

$$i \longrightarrow\!\!\rightsquigarrow (e_u, \theta_{g_u}) \ \nearrow \searrow \ \nwarrow \ \ldots$$

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$$i \xrightarrow{w_{(e,\pi)}} \rightsquigarrow (e, \pi) \begin{smallmatrix} \nearrow & \searrow \\ \nwarrow & \dots \end{smallmatrix}$$

6 ::: Preliminares

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$$\forall g\,,\; g \cdot e = e \;\Longrightarrow\; G \approx \{e\} \times G \hookrightarrow E *_\theta G$$

- E finite:

  $E *_\theta G$ Howson inverse semigroup $\Longleftrightarrow$ G Howson group.

$a_x \in A$ $(x \in X \subseteq E *_\theta G)$:

$$i \xrightarrow{a_x} (e_x, \theta_{g_x}) \;\text{ and }\; (f, \pi) \xrightarrow{a_x} (f \wedge \pi(e_x), \pi\theta_{g_x})$$

$u \in \langle X_1 \rangle \cap \langle X_2 \rangle$: $u = (e_u, g_u) = (e_u, h)(h^{-1} \cdot e_u, h^{-1} g_u)$

$$i \xrightarrow{w_{(e,\pi)}} (e_u, \theta_h) \nearrow \searrow \ldots \nwarrow$$

9 ::: Main result

Setting

action $\theta\colon G \to \mathrm{Aut}(E)$ is **locally finite** if

$$H \cdot e = \{h \cdot e \colon h \in H\}$$

is finite for any $H \leq_{\text{f.g.}} G$ and any $e \in E$.

- $\theta\colon G \to \mathrm{Aut}(E)$ locally finite action:
  $E *_\theta G$ Howson inverse semigroup $\iff$ G Howson group.

Idea of proof: Reduction to the finite case

$E *_\theta G$ **Howson inverse semigroup** $\implies$ G **Howson group:**

$$
\begin{aligned}
H_1, H_2 \leq_{\text{f.g.}} G &\implies G' = \langle H_1 \cup H_2 \rangle \leq_{\text{f.g.}} G \\
&\implies G' \cdot e \text{ is finite} && (\theta \text{ locally finite}) \\
&\implies E' = \left\langle G' \cdot e \right\rangle \text{ is finite} && (E' \text{ (sub)semilattice}) \\
&\implies E' *_\theta G' \hookrightarrow E *_\theta G \\
&\implies G' \text{ Howson} \\
&\implies H_1 \cap H_2 \text{ f.g.}
\end{aligned}
$$

G **Howson group** $\implies$ $E *_\theta G$ **Howson inverse semigroup:**

...

11 ::: Idea of proof: Reduction to the finite case, cont.

$E *_\theta G$ **Howson inverse semigroup** $\implies$ G **Howson group:**

...

G **Howson group** $\implies$ $E *_\theta G$ **Howson inverse semigroup:**

$$
\begin{aligned}
&X_1, X_2 \subseteq E *_\theta G \text{ finite } \implies \\
&\implies G' = \langle \pi_G(x) \colon x \in X_1 \cup X_2 \rangle \leq_{\text{f.g.}} G \\
&\implies E' = \left\langle G' \cdot \pi_E(x) \colon x \in X_1 \cup X_2 \right\rangle \text{ is finite} \quad (\theta \text{ locally finite}) \\
&\implies E' *_\theta G' \text{ Howson} \qquad (G' \text{ Howson}) \\
&\implies \langle X_1 \rangle \cap \langle X_2 \rangle \text{ f.g.}
\end{aligned}
$$

12 ::: Some consequences, some examples

- G Howson group $\implies$ $E \times G$ Howson inverse semigroup.

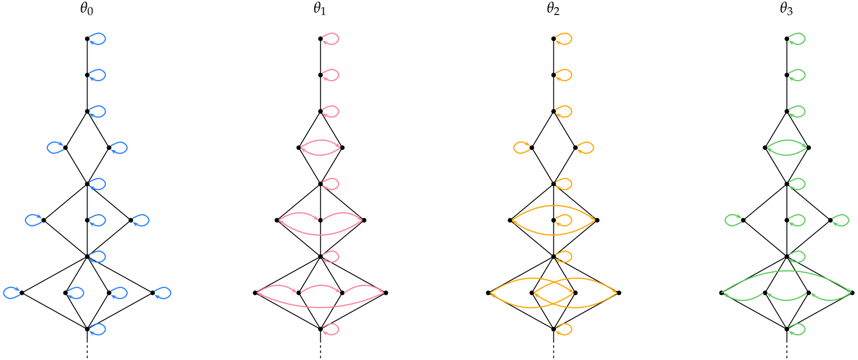
- G locally finite group $\implies$ $E *_\theta G$ Howson inverse semigroup.

- $E = \{(2n+1, 0) \colon n \geq 0\} \cup (\bigcup_{n \geq 1} \{2n\} \times \mathbb{Z}/n\mathbb{Z})$
  with $(k, x) \leq (\ell, y)$ iff $k > \ell$ or $(k, x) = (\ell, y)$
- $G = (\mathbb{Z}, +)$ (Howson, but not locally finite)
- $\theta_m(2n+1, 0) = (2n+1, 0)$ & $\theta_m(2n, k + n\mathbb{Z}) = (2n, m + k + n\mathbb{Z})$
  $\implies$ locally finite action
  $\implies$ $E *_\theta G$ is a Howson inverse semigroup.

13 ::: Locally finite action of a non locally finite group

$m \in \mathbb{Z}$,

$\theta_m(2n+1, 0) = (2n+1, 0)$ & $\theta_m(2n, k + n\mathbb{Z}) = (2n, m + k + n\mathbb{Z})$



14 ::: Some consequences, some examples, cont.

- FIS_n free inverse semigroup, $FIS_n \hookrightarrow E *_\theta G$;
  $n \geq 2 \implies E *_\theta G$ not Howson inv. smg. (but G Howson)
  $\implies \theta$ not locally finite.

- $E = (\mathbb{Z}, \leq)$
- $G = (\mathbb{Z}, +)$ (Howson)
- $\theta_n(m) = n + m$ (not locally finte)
  $\implies E *_\theta G$ Howson inverse semigroup.

G not Howson?

15 ::: On the bound

Howson/H. Neumann/Friedman/Mineyev. F free group, $H_1, H_2 \leq_{f.g.} F$

$$\implies \mathrm{rk}(H_1 \cap H_2) \leq (\mathrm{rk}(H_1) - 1)(\mathrm{rk}(H_2) - 1) + 1.$$

S is **polynomially Howson** if there exists a polynomial $p(x) \in \mathbb{R}[x]$ such that $\forall\, T_1, T_2 \leq S$

$$\mathrm{rk}(T_1), \mathrm{rk}(T_2) \leq n \implies \mathrm{rk}(T_1 \cap T_2) \leq p(n)\,.$$

- E be a finite semilattice
- G polynomially Howson

$$\implies E *_\theta G \text{ polynomially Howson.}$$

16 ::: Some references

- **Howson (1954):** On the intersection of finitely generated free groups.
- **Jones & Trotter (1989):** The Howson property for free inverse semigroups.
- **Margolis & Meakin (1993):** Free inverse semigroups and graph immersions.
- **Araújo & Silva & Sykiotis (2014):** Finiteness results for subgroups of finite extensions.
- ...

- **Silva & FS (to appear):** Howson's property for semidirect products of semilattices by groups.