Torelli type results for logarithmic bundles of arrangements in $\mathbf{P}^n$

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PRELIMINARIES

- Multi-degree arrangements with normal crossings in $\mathbb{P}^n$
- Logarithmic bundles and Torelli type questions
- Some known results
- Related topics

RECENT RESULTS

- Multi-degree arrangements with *many* objects
- The conic case
- Some line-conic cases

OPEN PROBLEMS

Let $\mathbb{P}^n$ be the n-dimensional complex projective space with $n \geq 2$.

Definition

A *multi-degree arrangement* on $\mathbb{P}^n$, with degrees $d_1, \dots, d_\ell$, is a family

$$\mathcal{D} = \{D_1, \dots, D_\ell\}$$

of smooth irreducible hypersurfaces of $\mathbb{P}^n$ such that $D_i \neq D_j$.

Definition

A multi-degree arrangement $\mathcal{D}$ on $\mathbb{P}^n$ has *normal crossings* if it is locally isomorphic to a union of coordinate hyperplanes of $\mathbb{C}^n$.

Assume that $D_i = \{f_i = 0\}$ with $f_i \in \mathbb{C}[x_0, \dots, x_n]_{d_i}$.

Let $\mathbb{P}^n$ be the n-dimensional complex projective space with $n \geq 2$.

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Assume that $D_i = \{f_i = 0\}$ with $f_i \in \mathbb{C}[x_0, \ldots, x_n]_{d_i}$.

Let $\mathcal{D}$ be a multi-degree arrangement with normal crossings on $\mathbb{P}^n$ and let $\jmath$ be the embedding of $\mathbb{P}^n - \mathcal{D}$ in $\mathbb{P}^n$.

Definition (Deligne 1971, [De])

The *sheaf of differential 1-forms on* $\mathbb{P}^n$ *with logarithmic poles on* $\mathcal{D}$ is the subsheaf $\Omega^1_{\mathbb{P}^n}(\log \mathcal{D})$ of $\jmath_* \Omega^1_{\mathbb{P}^n - \mathcal{D}}$ s.t. $\Gamma(I_x, \Omega^1_{\mathbb{P}^n}(\log \mathcal{D}))$ is

$$\left\{ s \in \Gamma(I_x, \jmath_* \Omega^1_{\mathbb{P}^n - \mathcal{D}}) \,\middle|\, s = \sum_{i=1}^{k} u_i d \log z_i + \sum_{i=k+1}^{n} v_i dz_i \right\}$$

where $x \in \mathbb{P}^n$ and $I_x \cap \mathcal{D} = \{z_1 \cdot \ldots \cdot z_k = 0\}$.

Proposition

$\Omega^1_{\mathbb{P}^n}(\log \mathcal{D})$ is a vector bundle of rank n with the exact sequences

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n} \longrightarrow \Omega^1_{\mathbb{P}^n}(\log \mathcal{D}) \xrightarrow{\text{res}} \bigoplus_{i=1}^{\ell} \mathcal{O}_{D_i} \longrightarrow 0$$

where res is the *Poincaré residue morphism* and ([A2])

$$0 \longrightarrow \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbb{P}^n}(-d_i)^{\ell} \xrightarrow{\text{M}} \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \oplus \mathcal{O}_{\mathbb{P}^n}^{\ell-1} \longrightarrow \Omega^1_{\mathbb{P}^n}(\log \mathcal{D}) \longrightarrow 0$$

where

$$\text{M} = \begin{pmatrix} \nabla f_1 & f_1 & 0 & \cdots & 0 \\ \nabla f_2 & 0 & f_2 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ \nabla f_{\ell-1} & 0 & \cdots & 0 & f_{\ell-1} \\ \nabla f_\ell & 0 & \cdots & \cdots & 0 \end{pmatrix}^T .$$

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Torelli type questions

Let us consider the correspondence:

$$\mathcal{D} \longmapsto \Omega^1_{\mathbb{P}^n}(\log \mathcal{D}).$$

Is it *injective*? This is the *Torelli problem* for logarithmic bundles. In the case of 1:1 correspondance, $\mathcal{D}$ is called of *Torelli type*.

Definition

A hypersurface $D \subset \mathbb{P}^n$ is *unstable* for $\Omega^1_{\mathbb{P}^n}(\log \mathcal{D})$ if:

$$H^0(D, \Omega^1_{\mathbb{P}^n}(\log \mathcal{D})^\vee_{|D}) \neq \{0\}.$$

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Theorem (Dolgachev Kapranov 1993, [DK] - Vallès 2000, [V])

Let $\mathcal{H} = \{H_1, \dots, H_\ell\}$ be a n.c. hyperplane arrangement on $\mathbb{P}^n$.

- $\ell \leq n+2 \Longrightarrow \mathcal{H}$ is not of Torelli type
- $\ell \geq n+3 \Longrightarrow \mathcal{H}$ is the set of unst. hyperp. of $\Omega^1_{\mathbb{P}^n}(\log \mathcal{H})$, unless $H_1, \dots, H_\ell$ osculate a r.n.c. $\mathcal{C}_n \subset \mathbb{P}^n$ of degree n, in which case all the hyperp. "lying" on $\mathcal{C}_n^\vee \subset (\mathbb{P}^n)^\vee$ are unstable and $\Omega^1_{\mathbb{P}^n}(\log \mathcal{H}) \cong E_{\ell-2}(\mathcal{C}_n^\vee)$.

Further results have been proved by Ancona and Ottaviani, [AO].

Theorem (Ueda-Yoshinaga 2009, [UY2])

Let D be a general hypersurface of degree d in $\mathbb{P}^n$. $\mathcal{D} = \{D\}$ is of Torelli type if and only if $d \geq 3$.

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What happens if the arrangement has not n.c.?

- 2013: Faenzi, Matei and Vallès proved that a hyperplane arrangement $\mathcal{H}$ is of Torelli type $\Longleftrightarrow H_1, \dots, H_\ell \notin$ *Kronecker-Weierstrass variety*, [FMV]
- 2014: Generalizing the technique of U-Y, Dimca and Sernesi proved a Torelli type theorem for one singular curve in $\mathbb{P}^2$ with few nodes and cusps, [DS].

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Let $\mathcal{D} = \{D_1^{d_1}, \ldots, D_{\ell_1}^{d_1}, \ldots\ldots, D_1^{d_m}, \ldots, D_{\ell_m}^{d_m}\}$ be a multi-degree arrangement with normal crossings on $\mathbb{P}^n$ s.t. $D_1^{d_i}, \ldots, D_{\ell_i}^{d_i}$ have degree d_i and $d_m > d_{m-1} > \cdots > d_1$. Let $N_i = \binom{n+d_i}{d_i} - 1$.

Hypersurfaces - Hyperplane sections correspondence, [H]

Let $\nu_{d_i} : \mathbb{P}^n \longrightarrow \mathbb{P}^{N_i}$ be the d_i-uple Veronese embedding. Then:

$$\{D_1^{d_i}, \ldots, D_{\ell_i}^{d_i}\} \longrightarrow \mathcal{H}_{d_i} = \{H_1^{d_i}, \ldots, H_\ell^{d_i}\}$$

Let $\nu : \mathbb{P}^n \longrightarrow \mathbb{P} = \prod_{i=1}^m \mathbb{P}^{N_{d_i}}$ be the diagonal embedding

$$\nu([x_0, \ldots, x_n]) = [\nu_{d_1}([x_0, \ldots, x_n]), \ldots, \nu_{d_m}([x_0, \ldots, x_n])]$$

and $p_i : \mathbb{P} \longrightarrow \mathbb{P}^{N_i}$ the i-th projection. Then:

$$\mathcal{D} \longrightarrow \mathcal{A} = \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_m$$

with $\mathcal{A}_i = p_i^*(\mathcal{H}_{d_i})$.

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Torelli theorem for multi-degree arrangements with *many* objects (A. 2014, [A3])

Let $\mathcal{D}$ be a multi-degree arrangement with normal crossings on $\mathbb{P}^n$. Let $\mathcal{H}^{d_1}, \ldots, \mathcal{H}^{d_m}$ and $\mathcal{A}$ be the corresponding arrangements on $\mathbb{P}^{N_{d_1}}, \ldots, \mathbb{P}^{N_{d_m}}$ and $\mathbb{P}$. Assume that, for all $i \in \{1, \ldots, m\}$:

- $\ell_i \geq N_{d_i} + 4$
- $\mathcal{A}$ has normal crossings on $\mathbb{P}$
- $\mathcal{H}_{d_i}$ has normal crossings on $\mathbb{P}^{N_{d_i}}$ and its hyperplanes don't osculate a r.n.c. of degree N_{d_i} in $\mathbb{P}^{N_{d_i}}$.

Then: $\mathcal{D} = \{$smooth irr. degree-d_i hypers. unst. for $\Omega^1_{\mathbb{P}^n}(\log \mathcal{D})\}$.

Corollary

$\ell_i \geq \binom{n+d_i}{d_i} + 3, \forall i \in \{1, \ldots, m\} \implies$ the map is gen. 1:1.

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Proof.

- $D \in \mathcal{D} \Longrightarrow D$ unstable for $\Omega^1_{\mathbb{P}^n}(\log \mathcal{D})$
- D smooth irreducible unstable for $\Omega^1_{\mathbb{P}^n}(\log \mathcal{D})$, $\deg D = d_i$
  1) $i = m$: show H_{d_m} unstable hyperplane for $\Omega^1_{\mathbb{P}^{N_m}}(\log \mathcal{H}_{d_m})$
     $\overset{\text{DKV}}{\Longrightarrow} H_{d_m} \in \mathcal{H}_{d_m} \Longrightarrow D = D_j^{d_m} \in \mathcal{D}$
  2) $i \in \{m-1, m-2, \ldots, 1\}$: *reduce* $\Omega^1_{\mathbb{P}^n}(\log \mathcal{D})$ by the hypersurfaces of $\mathcal{D}$ of degree d_m and iterate
     e.g.: $0 \to \Omega^1_{\mathbb{P}^n}(\log(\mathcal{D} - \{D^{d_m}_{\ell_m}\})) \to \Omega^1_{\mathbb{P}^n}(\log \mathcal{D}) \to \mathcal{O}_{D^{d_m}_{\ell_m}} \to 0$

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Let $\mathcal{D} = \{C_1, \ldots, C_\ell\}$ be a conic arrangement with n.c. on $\mathbb{P}^2$.

Proposition (A. 2014, [A2])

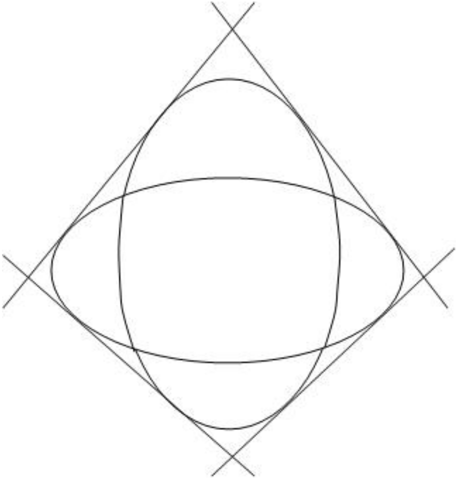
$\ell = 1 \Longrightarrow \Omega^1_{\mathbb{P}^2}(\log \mathcal{D}) \cong T\mathbb{P}^2(-2)$.

Theorem (A. 2014, [A2])

$\ell = 2 \Longrightarrow$ the isomorphism class of $\Omega^1_{\mathbb{P}^n}(\log \mathcal{D})$ is determined by the four bitangent lines to the pair of conics.

Remark

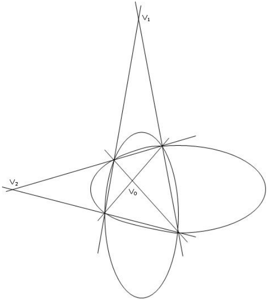
The previous results can be extended to the quadric case in $\mathbb{P}^n$.



Four bitangent lines to a pair of conics

The proof of this theorem is based on the following facts:

- $\mathcal{D} = \{C_1, C_2\}$ has n.c. $\iff$ in the pencil generated by C_1 and C_2 there are 3 singular conics (in this case C_1 and C_2 are *simultaneously diagonalizable* in the "basis" $\{v_0, v_1, v_2\}$).
- $\{v_0, v_1, v_2\}$ is the zero locus of the section of $\Omega^1_{\mathbb{P}^2}(\log \mathcal{D})$.



Two conics with normal crossings

Proof.

$\Rightarrow$ $\Omega^1_{\mathbb{P}^2}(\log \mathcal{D}) \cong \Omega^1_{\mathbb{P}^2}(\log \mathcal{D}') \to \{v_0, v_1, v_2\} = \{v_0', v_1', v_2'\} = \mathcal{B}$
With respect to $\mathcal{B}$, we can represent C_1, C_2, C_1', C_2' with diagonal matrices, say A, B, C, D. Then:

$$C^{-1} = (A^{-1} + tB^{-1}) \text{ and } D^{-1} = (A^{-1} + sB^{-1}).$$

$\Leftarrow$ $\mathcal{D}$ has normal crossings $\to$ A, B diagonal matrices
$C_1^\vee \cap C_2^\vee = C_1'^\vee \cap C_2'^\vee \to C$, D as above.

□

Theorem (A. 2015, [A3])

If $\ell \geq 4$ then the map
$$\mathcal{D} = \{C_1, C_2, C_3, C_4\} \longrightarrow \Omega^1_{\mathbb{P}^2}(\log \mathcal{D})$$
is generically injective.

Proof.

1) Assume $\ell = 4$. With M2, get a $\mathcal{D}$ with 4 random conics such that $\Omega^1_{\mathbb{P}^2}(\log \mathcal{D})$ has 4 unstable conics.
2) Find an open set $V \subset \mathbb{P}_5 \times \mathbb{P}_5 \times \mathbb{P}_5 \times \mathbb{P}_5$ s.t., $\forall \mathcal{D} \in V$, $\Omega^1_{\mathbb{P}^2}(\log \mathcal{D})$ has 4 unstable conics.
3) If $\ell \geq 5$ apply the *reduction technique*.

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$\underline{\ell = 3}$

We recall that $\Omega^1_{\mathbb{P}^2}(\log \mathcal{D})$ admits the exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2)^3 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^3 \oplus \mathcal{O}^2_{\mathbb{P}^2} \longrightarrow \Omega^1_{\mathbb{P}^2}(\log \mathcal{D}) \longrightarrow 0$$

$$\Downarrow$$

- ⋆ $\Omega^1_{\mathbb{P}^2}(\log \mathcal{D})$ lives in $\mathcal{M}_{\mathbb{P}^2}(-1, 7)$, a moduli space of dimension 24 while three conics give 15 parameters
- ⋆ For an element of $\mathcal{M}_{\mathbb{P}^2}(-1, 7)$ we expect 21 unstable conics: in some examples with M2 we get this fact, but it seems to be hard to find these conics
- ⋆ For an element of $\mathcal{M}_{\mathbb{P}^2}(-1, 7)$ we expect 21 unstable lines: in some examples with M2 we get this fact, we are able to find such lines but it's quite difficult to interpret them...

$\underline{\ell = 3}$

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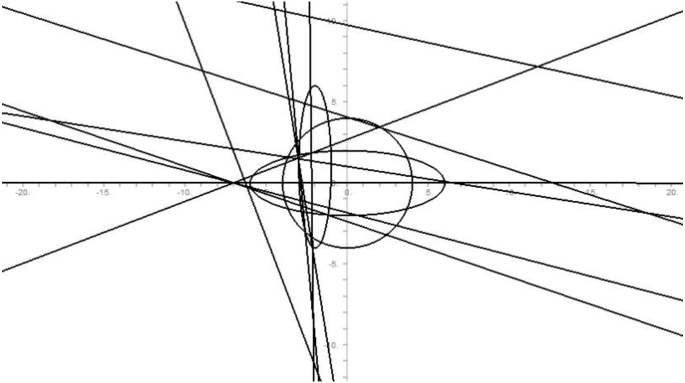
$$\Downarrow$$

- $\star$ $\Omega^1_{\mathbb{P}^2}(\log \mathcal{D})$ lives in $\mathcal{M}_{\mathbb{P}^2}(-1, 7)$, a moduli space of dimension 24 while three conics give 15 parameters
- $\star$ For an element of $\mathcal{M}_{\mathbb{P}^2}(-1, 7)$ we expect 21 unstable conics: in some examples with M2 we get this fact, but it seems to be hard to find these conics
- $\star$ For an element of $\mathcal{M}_{\mathbb{P}^2}(-1, 7)$ we expect 21 unstable lines: in some examples with M2 we get this fact, we are able to find such lines but it's quite difficult to interpret them...

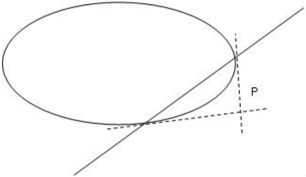


11 real unstable lines for 3 conics

Let $\mathcal{D} = \{r_1, \ldots, r_\ell, C\}$ be a n.c. line-conic arrangement on $\mathbb{P}^2$.

Theorem (A. 2015, [A3])

$\ell = 1 \Longrightarrow$ the isomorphism class of $\Omega^1_{\mathbb{P}^2}(\log \mathcal{D})$ is determined by the pole of the line with respect to the conic.

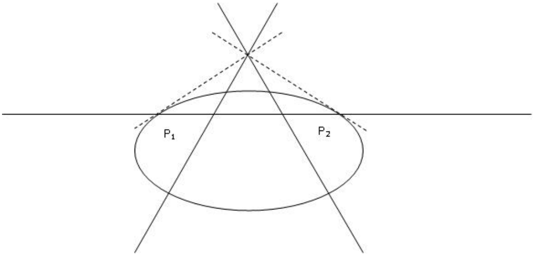


A conic and a line

This result can be extended to $\mathbb{P}^n$, $n \geq 3$.

Theorem (A. 2015, [A3])

$\ell = 2 \Longrightarrow$ the isomorphism class of $\Omega^1_{\mathbb{P}^2}(\log \mathcal{D})$ is determined by the points $\{P_1, P_2\}$ as in the picture.



A conic and two lines

Theorem (A. 2015, [A3])

$\ell = 3 \Longrightarrow \exists$ a smooth cubic $D \subset \mathbb{P}^2$ s.t. if $\mathcal{D}' = \{D\}$ then

$$\Omega^1_{\mathbb{P}^2}(\log \mathcal{D}) \cong \Omega^1_{\mathbb{P}^2}(\log \mathcal{D}')(1).$$

Open problem

We know that a Torelli type theorem doesn't hold, but can we say more about the fiber of our map?

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Open problems

- $\mathbb{P}^2$
  - Get a complete theorem in the line-conic case
  - Study arrangements with *few* objects of degree ≥ 3
  - Study *multi-degree* arrangements with *few* objects
  - ...
- Generalize to $\mathbb{P}^n$

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