Nonstandard Intuitionistic Interpretations

B. Dinis[1] F. Ferreira[2]

[1]Universidade de Lisboa - CMAF
CIMA

[2]Universidade de Lisboa

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Outline

1. $E\text{-}HA^{\omega}_{st}$
   - Majorizability
   - Axioms
2. Realizability
   - $\tilde{\exists}^{st}$-free formulas.
   - Realizability
   - Characteristic principles
3. Intuitionistic Nonstandard Functional Interpretation
   - The interpretation
   - Characteristic Principles
   - Main results
4. *WKL*

Goal: Study constructive proofs and extract additional information from them.

- Formulas are "realized" by objects (realizers), so that the knowledge of the realizer gives knowledge about the truth of the formula.
- Realizability can be seen as a formalization of the BHK interpretation of intuitionistic logic. The notion of "proof"is replaced with a formal notion of "realizer".
- Bounded functional interpretations, can be thought of as modifications of the usual functional interpretations where bounds (rather than precise witnesses) are extracted from proofs.

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Majorizability

In the language L^{ω}, the Howard/Bezem notion of (strong) majorizability is defined by:

- $x \leq^*_0 y$ is $x \leq y$
- $x \leq^*_{\rho\to\sigma} y$ is $\forall v \forall u \leq^*_\rho v\,(xu \leq^*_\sigma yv \wedge yu \leq^*_\sigma yv)$

Notation: $\tilde{\forall} x \Phi(x)$ abbreviates $\forall x\,(x \leq^* x \to \Phi(x))$
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- A formula is called internal if it is part of the original language of E-HA$^{\omega}$

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- and it is called external otherwise.
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$\text{E-HA}^{\omega}_{\text{st}}$ axioms

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Standardness axioms (ST):

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2. $\mathrm{st}^{\sigma}(y) \to (x \leq^{*}_{\sigma} y \to \mathrm{st}^{\sigma}(x))$;
3. $\mathrm{st}^{\sigma}(t)$, for each closed term t of type σ;
4. $\mathrm{st}^{\sigma\to\tau}(z) \to (\mathrm{st}^{\sigma}(x) \to \mathrm{st}^{\tau}(zx))$;

   where the types σ and τ are arbitrary.

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External induction rule (EI):

- From $\Phi(0)$ and $\forall^{\text{st}} n^0 (\Phi(n) \to \Phi(n+1))$, infer $\forall^{\text{st}} n^0 \Phi(n)$.

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$\tilde{\exists}^{\mathrm{st}}$-free formulas

A formula Φ is called $\tilde{\exists}^{\mathrm{st}}$-free if it is built from atomic internal formulas by means of $\wedge$, $\vee$, $\rightarrow$, $\forall$, $\exists$, and $\tilde{\forall}$.

Realizability

To each formula Φ we assign formulas Φ^{bs} and Φ_{bs} so that Φ^{bs} is of the form $\tilde{\exists}^{\mathrm{st}} c\, \Phi_{\mathrm{bs}}(c)$, with $\Phi_{\mathrm{bs}}(c)$ $\tilde{\exists}^{\mathrm{st}}$-free, according to:

1. Φ^{bs} and Φ_{bs} are simply Φ, for internal atomic formulas Φ,
2. $\mathrm{st}(t)^{\mathrm{bs}}$ is $\tilde{\exists}^{\mathrm{st}} c[t \leq^* c]$.
3. $(\Phi \wedge \Psi)^{\mathrm{bs}}$ is $\tilde{\exists}^{\mathrm{st}} c, d\, [\Phi_{\mathrm{bs}}(c) \wedge \Psi_{\mathrm{bs}}(d)]$,
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Theorem

Let Φ be an $\tilde{\exists}^{\mathrm{st}}$-free formula. Then $(\Phi)^{bs}$ is Φ_{bs} and they are both equivalent to Φ.

Proof.

By induction on the complexity of the $\tilde{\exists}^{\mathrm{st}}$-free formula Φ. □

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Characteristic principles

I. mAC^{ω}: $\tilde{\forall}^{\mathrm{st}} x \tilde{\exists}^{\mathrm{st}} y \, \Phi(x, y) \to \tilde{\exists}^{\mathrm{st}} f \tilde{\forall}^{\mathrm{st}} x \tilde{\exists} y \leq^* fx \, \Phi(x, y)$.

II. R^{ω}: $\forall x \exists^{\mathrm{st}} y \, \Phi(x, y) \to \tilde{\exists}^{\mathrm{st}} z \forall x \exists y \leq^* z \, \Phi(x, y)$.

III. $\mathrm{IP}^{\omega}_{\tilde{\exists}^{\mathrm{st}}\text{-free}}$: $\left(A \to \tilde{\exists}^{\mathrm{st}} y \, \Psi(y)\right) \to \tilde{\exists}^{\mathrm{st}} y \left(A \to \tilde{\exists} z \leq^* y \, \Psi(z)\right)$,
where A is an $\tilde{\exists}^{\mathrm{st}}$-free formula.

IV. MAJ^{ω}: $\forall^{\mathrm{st}} x \exists^{\mathrm{st}} y (x \leq^* y)$.

Characteristic principles

I. mAC$^\omega$: $\tilde{\forall}^{\mathrm{st}} x \tilde{\exists}^{\mathrm{st}} y\, \Phi(x,y) \rightarrow \tilde{\exists}^{\mathrm{st}} f \tilde{\forall}^{\mathrm{st}} x \tilde{\exists} y \leq^* fx\, \Phi(x,y)$.

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Theorem (Soundness)

Suppose that

- $\text{E-HA}^{\omega}_{\text{st}} + \text{mAC}^{\omega} + \text{R}^{\omega} + \text{IP}^{\omega}_{\tilde{\exists}^{\text{st}}\text{-free}} + \text{MAJ}^{\omega} \vdash \Phi(z)$,

  Then there are closed monotone terms t of appropriate types such that
- $\text{E-HA}^{\omega}_{\text{st}} \vdash \Phi_{\text{bs}}(z, t)$.

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Theorem (Characterization)

$$\text{E-HA}^{\omega}_{\text{st}} + \text{mAC}^{\omega} + \text{R}^{\omega} + \text{IP}^{\omega}_{\tilde{\exists}^{\text{st}}\text{-}free} + \text{MAJ}^{\omega} \vdash \Phi^{\text{bs}} \leftrightarrow \Phi.$$

Outline

To each formula Φ assign formulas $\Phi^{\mathrm{I_{st}}}$ and $\Phi_{\mathrm{I_{st}}}$ so that $\Phi^{\mathrm{I_{st}}}$ is of the form $\tilde{\exists}^{\mathrm{st}} b \tilde{\forall}^{\mathrm{st}} c\, \varphi_{\mathrm{I_{st}}}(b, c)$, according to:

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Characteristic Principles

I. mAC$^{\omega}$: $\tilde{\forall}^{\mathrm{st}} x \tilde{\exists}^{\mathrm{st}} y \, \Phi(x,y) \to \tilde{\exists}^{\mathrm{st}} f \tilde{\forall}^{\mathrm{st}} x \tilde{\exists} y \leq^{*} fx \, \Phi(x,y)$.

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I. mAC^{ω}: $\tilde{\forall}^{\mathrm{st}} x \tilde{\exists}^{\mathrm{st}} y\, \Phi(x,y) \to \tilde{\exists}^{\mathrm{st}} f \tilde{\forall}^{\mathrm{st}} x \tilde{\exists} y \leq^{*} fx\, \Phi(x,y)$.

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III. I^{ω}: $\tilde{\forall}^{\mathrm{st}} z \exists x \forall y \leq^{*} z\, \phi(x,y) \to \exists x \forall^{\mathrm{st}} y\, \phi(x,y)$.

IV. $\mathrm{IP}^{\omega}_{\tilde{\forall}^{\mathrm{st}}}$: $\left(\Phi \to \tilde{\exists}^{\mathrm{st}} y\, \Psi(y)\right) \to \tilde{\exists}^{\mathrm{st}} y \left(\Phi \to \tilde{\exists} z \leq^{*} y\, \Psi(z)\right)$,

where Φ is of the form $\tilde{\forall}^{\mathrm{st}} x\, \phi(x,y)$.

V. $\mathrm{MP}^{\mathrm{st}}$: $\left(\tilde{\forall}^{\mathrm{st}} x\, \phi(x) \to \psi\right) \to \tilde{\exists}^{\mathrm{st}} y \left(\tilde{\forall} x \leq^{*} y\, \phi(x) \to \psi\right)$.

VI. MAJ^{ω}: $\forall^{\mathrm{st}} x \exists^{\mathrm{st}} y (x \leq^{*} y)$.

Soundness

Theorem (Soundness)

Suppose that

- $\text{E-HA}^{\omega}_{\text{st}} + \text{mAC}^{\omega} + \text{R}^{\omega} + \text{I}^{\omega} + \text{MP}^{\text{st}} + \text{IP}^{\omega}_{\tilde{\forall}^{\text{st}}} + \text{MAJ}^{\omega} \vdash \Phi$,

  Then there are closed monotone terms t of appropriate types such that
- $\text{E-HA}^{\omega}_{\text{st}} \vdash \tilde{\forall}^{\text{st}} b\, \Phi_{\text{bs}}(b, t)$.

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Let Φ be an arbitrary formula (possibly with free variables). Then

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Outline

WKL

Weak König's lemma can be stated as follows: For every binary tree T,

$$\forall x \leq_1 1 \exists n \left(x\left(n\right) \notin T\right) \rightarrow \exists n \forall x \leq_1 1 \left(x\left(n\right) \notin T\right).$$

Removing WKL

- Binary bar recursion (Howard, Troelstra)
- Monotone functional interpretation (Kohlenbach)
- Bounded functional interpretation (Ferreira, Gaspar)
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Interpretation of *WKL* in the theory of Berg et al.

$$L^2 \rightsquigarrow L^{\omega^*}_{st}$$

- first-order variables $\rightsquigarrow$ variables of type 0 satisfying *st*.
- second-order variables $\rightsquigarrow$ variables of type 0.
- $x \in X \rightsquigarrow p_x | X$

Second-order variables are ranging in the *Scott system* associated with the (type 0) variables of the theory.

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