Distribution of implicit interest rate for microcredit

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What is Microcredit?

- Very small loans to economically disadvantaged people
- Large number of (frequent) installments leading to little default risk but possible delay in installments

Yunus' example

- Loan: 1000 BDT[a]
- repaied in 50 installments of 22BDT paied in week k, so at time $t_k = k$.

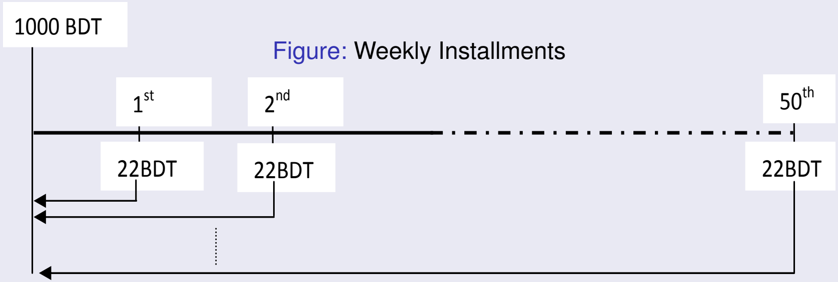


Figure: Weekly Installments

[a] One EUR is a little less then hundred BDT

Yunus' equation: defines the implicit yearly interest rate r^Y

If we denote by r^Y the implicit yearly interest rate, we see that this "Yunus" interest rate r^Y is implicitly defined by the equation

$$1000 = \sum_{k=1}^{50} 22 e^{-t_k \frac{r^Y}{52}} \quad (1)$$

$$= 22 \sum_{k=1}^{50} q^k = 22 \frac{q - q^{51}}{1 - q} \text{ as } t_k = k, \text{ with } q = e^{-\frac{r^Y}{52}}. \quad (2)$$

Numerical solution: $q = 0.9962107\ldots$ and
$r^Y = -52 \log(q) = \mathbf{19.74\ldots\%} \simeq 20\%$, as given by Yunus.

Stochastic model for installments dates T_k

- Let $\mathcal{B} = (B_m)_{m \geq 1}$ be a Bernoulli process, $B_m \rightsquigarrow \mathcal{B}(1, p)$.
- $B_m = 1$ stands for “the borrower can pay at week m", and $B_m = 0$ models that she can’t pay at this week.
- The k-th installment takes place at (the $\mathcal{F}^{\mathcal{B}}$ stopping-time) $T_k = \min\{m | B_1 + \ldots + B_m = k\}$, $k = 1 \ldots N$.
- $X_k = T_k - T_{k-1}$ is the*time to k-th installment* , with $T_0 = 0$.
- We see that $\mathbf{P}(X = x) = p(1-p)^{x-1}$ so $X_k \rightsquigarrow \mathcal{G}(p)$, the geometric distribution.

$p = 84\%$

This gives a way to choose a reasonably realistic value of p. Indeed, if we take the often cited 3% of default-rate and if we consider[a] that default means that some $X_m > 4$, we get $p = 0.84$, value that we will use in our examples.

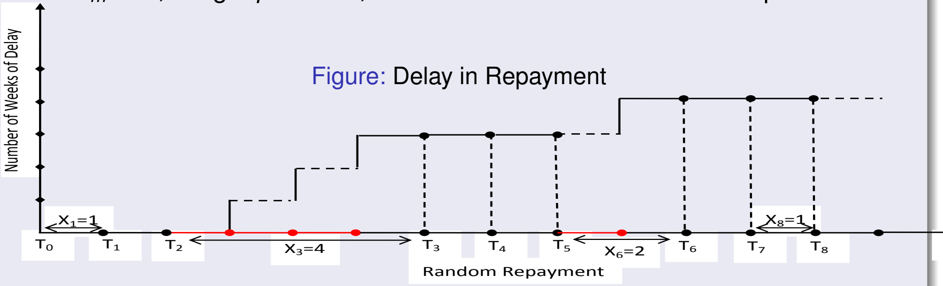


Figure: Delay in Repayment

[a] private communication of some BRAC credit-officer

The random implicit interest rate R

We also wish to see 50 as a large number of installments, so we replace 50 by N and 22 by $\frac{1000}{N}(1+r_f)$, where r_f is the *flat-rate* (10% in the case of Yunus' example.)

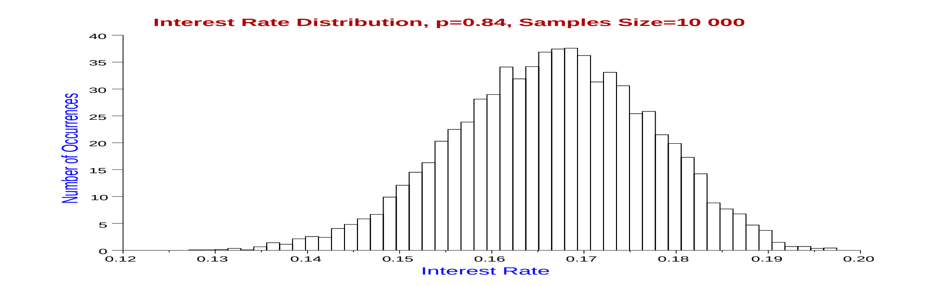
$$N = (1+r_f)\sum_{k=1}^{N} Q^{T_k} \text{ , with } Q = e^{-\frac{R_N}{N}} \tag{3}$$

We use here a capital letter R_N for the implicit interest rate, as, the T_ks being random, this interest rate becomes random too. We claim that this randomness is one of the main risk factors and we wish to better understand its probabilisty distribution.

Empirical probability distribution of the interest rate

Using `Scilab` we built a sample of size $10,000$ of random installments of lending.

Figure: Distribution of actuarial interest rate R_N computed from equation (3) for $N = 50$ and $r_f = 10\%$ on a Monte-Carlo sample of 10,000 borrowers.



Looking for the asymptotics of R

We will also take into account that the number (50) is large, so we assume that N is **infinitely large** fixed, and we just write R instead of R_N.
Looking at the histogram suggests, at first, a normal distribution. But it is obviously bounded from above, and the skewness is not zero (the hump is lopsided). We introduced the generalized Yunus equation because we expect to get an asymptotic expansion converging towards the distribution of R_N, i.e.

$$R_N = \sum_{l=0}^{m} \frac{\alpha_l}{N^l} + \frac{1}{N^n}\varepsilon$$

with m standard and $\varepsilon \simeq 0$.

Order 3 expansion of R^1

This program is far from being reached but we succeeded to get an asymptotic expansion for $n = 3$ of R^1_N, the implicit interest rate assuming a **single delay**, i.e. $T_N = N + 1$, or, equivalently, there is one and only one k such that $X_k = 2$, all other X_is being equal to 1.

The equation on Ω_1

Let $\Omega_1 = \{T_N = N+1\}$. On Ω_1 all events $\{X_k = 2\}$ have same probability, so $\mathbf{P}_{\Omega_1}(\{X_k = 2\}) = \frac{1}{N}$. Moreover, still on Ω_1, if $X_k = 2$, R_N^1 is determined and is the unique positive solution $r = r_k$ of

$$N = (1+r_f)\left(\sum_{m=1}^{k-1} q^m + \sum_{m=k}^{N} q^{m+1}\right), \text{ where } q = e^{-\frac{r}{N}} \tag{4}$$

which can be rewritten $N = (1+r_f)(\sum_{m=1}^{N+1} q^m - q^k)$ $\left(= (1+r_f)q[\frac{1-q^{N+1}}{1-q} - q^{k-1}]\right)$.

The theorem

Theorem

The unique positive solution q_k of (4) has the following asymptotic expansion

$$q_k = 1 - \frac{\beta_1}{N} + \frac{\beta_2}{N^2} + \frac{\beta_3(k)}{N^3} + \frac{\varepsilon(N)}{N^3}\text{, with } \lim_{N\to\infty} \varepsilon(N) = 0,$$

where β_1 *is the unique positive solution of* $\beta = (1 + r_f)(1 - e^{-\beta})$;
$\beta_2 = \frac{\beta_1^2(3+\beta_1-r_f)}{2(\beta_1-r_f)}$; $\beta_3 = \beta_3(k) = \lambda k + \mu$, *with* $\lambda = -\frac{\beta_1^2(1+r_f)}{\beta_1-r_f}$, *and* $\mu =$
$-\frac{\beta_1(1+r_f)}{\beta_1-r_f}\left[\frac{\beta_2^2}{\beta_1^2(1+r_f)} + \frac{1+r_f-\beta_1}{1+r_f}\left[\beta_2\left(\frac{3}{2} - \frac{\beta_2}{\beta_1^2} - \frac{\beta_2}{2\beta_1}\right) - \beta_1\left(1 + \frac{\beta_1}{2} - \frac{\beta_2}{2} + \frac{\beta_1^2}{8}\right)\right]\right]$.

Only the fourth term depends on k and is an **affine** function of k

When $r_f = 10\%$ $\beta_1 = 0.193748$. The striking fact here is that the three first terms of the expansion of q_k do not depend on k, and only the fourth, β_3 does, and is an **affine** function of k, $\beta_3(k) = \lambda k + \mu$. This is again so for the values of $r_k = -N\log(q_k)$; indeed expanding $\log(1-x)$ for small x we get

the interest rate depends linearly on the number of the week when the single delay occurs

Corollary

On Ω_1, when only one delay occurs in week k, the implicit interest rate r_k has an order two expansion

$$r_k = \alpha_0 + \frac{\alpha_1}{N} + \frac{\alpha_2(k)}{N^2} + \frac{\varepsilon(N)}{N^2}\text{ , with } \lim_{N\to\infty} \varepsilon(N) = 0.$$

with $\alpha_0 = \beta_1$, $\alpha_1 = \frac{1}{2}\beta_1^2 - \beta_2$, and $\alpha_2 = \alpha_2(k) = \frac{1}{3}\beta_1^3 - \frac{1}{2}\beta_1\beta_2 - \beta_3(k)$.

So, up to terms small with respect to $\frac{1}{N^2}$, the interest rate depends linearly on the number of the week when the single delay occurs. See figure 4.

R^1: asymptotic versus exact

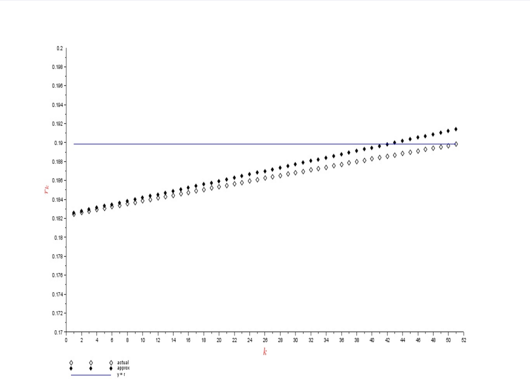
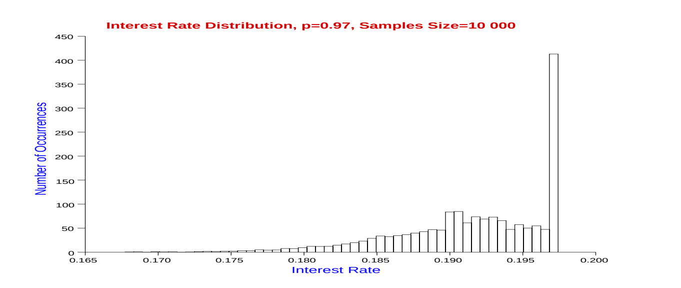


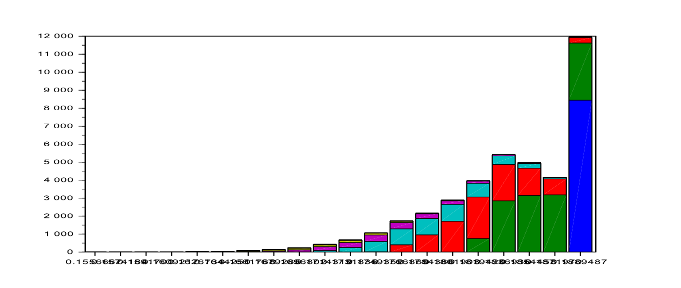
Figure: Actual interest rate in case of one single delay at week k and its asymptotic approximation. The horizontal line corresponds to the interest rate $r = r_{N+1}$, when there is no delay ($N = 50$, $r_f = 10\%$, $r^Y = \frac{52}{50}r$))

Distribution when $p = 97\%$



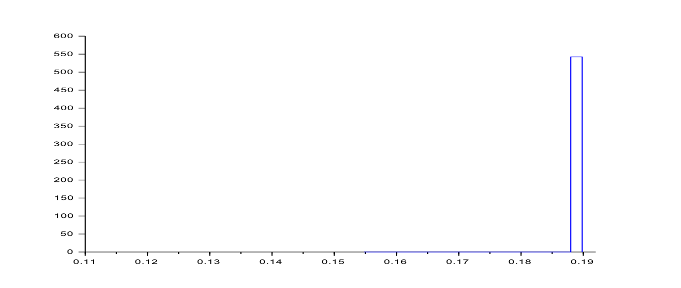
We assume now that the probability of in-time weekly installment p is 97%

Distribution when $p = 97\%$

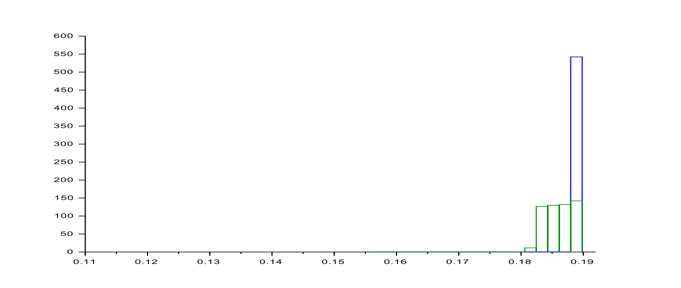


Various colors correspond to different total delay $T_N - N$.

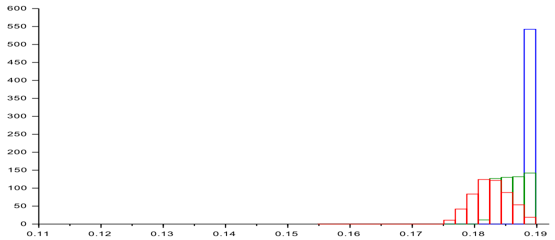
Distribution when $p = 97\%$ and Delay=$T_N - N = 0$



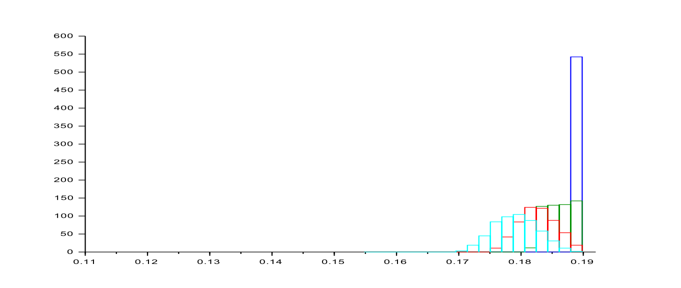
Distribution when $p = 97\%$ and Delay=$T_N - N = 0..1$



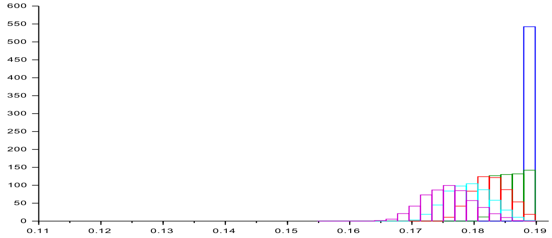
Distribution when $p = 97\%$ and Delay=$T_N - N = 0..2$



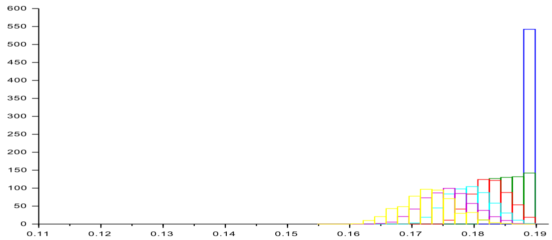
Distribution when $p = 97\%$ and Delay$=T_N - N = 0..3$



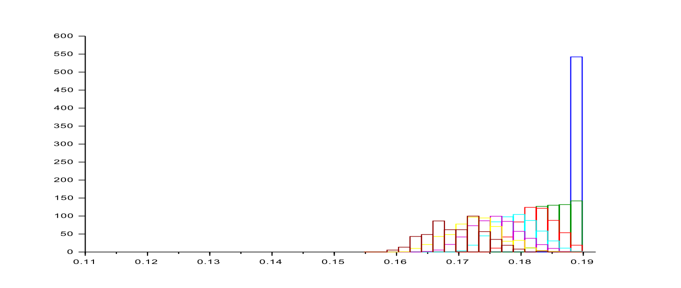
Distribution when $p = 97\%$ and Delay=$T_N - N = 0..4$



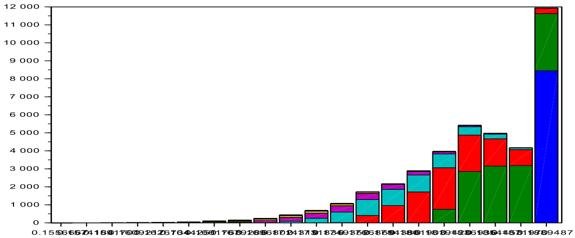
Distribution when $p = 97\%$ and Delay=$T_N - N = 0..5$



Distribution when $p = 97\%$ and Delay=$T_N - N = 0..6$

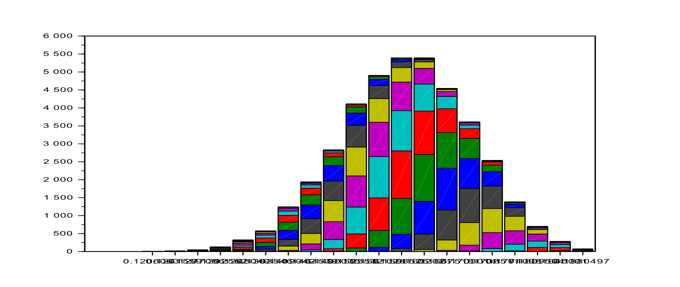


Here again: distribution when $p = 97\%$



Various colors correspond to different total delay $T_N - N$.

Distribution when $p = 84\%$



But if we assume now that the probability of in-time weekly installment p is 84% **we get a much more complicated picture**...

References

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Thanks

Thank you for your attention :-)