A Calderon-Zygmund decomposition for multiple frequencies and applications

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AMS - EMS - SPM International Meeting
Special Session - Geometric Aspects of Harmonic Analysis
June 11, 2015

Work being reported

- FDP and Christoph Thiele, *Endpoint bounds for the bilinear Hilbert transform*, to appear on Trans. Amer. Math. Soc., preprint arXiv:1403.5978
- FDP and Yumeng Ou (PhD Candidate, Brown U) *Banach valued multilinear singular integrals*, preprint available on personal webpage

Related work:

- C. Demeter, F. Di Plinio, *Endpoint bounds for the quartile operator*, J. Fourier Anal. Appl., **19**, (2013), no. 4, 836–856
- F. Di Plinio, *Weak-L^p bounds for the Carleson and Walsh-Carleson operators*, C. R. Math. Acad. Sci. Paris **352** (2014), no. 4, 327–331.
- F. Di Plinio and A. Lerner, *On weighted norm inequalities for the Carleson and Walsh-Carleson operators*, J. London Math. Soc. **90** (2014), no. 3, 654–674

What is a multifrequency CZ decomposition?

A schematic version:

- H Hilbert transform
- M the Hardy-Littlewood maximal function
- $\mathrm{Mod}_\xi f(x) = f(x)\exp(2\pi i x\xi)$ modulation by $\xi \in \mathbb{R}$.

Theorem

Let $\xi_1, \dots, \xi_N \in \mathbb{R}$. *Let* $f \in L^1(\mathbb{R})$, $\lambda > 0$,

$$E_\lambda = \{x \in \mathbb{R} : Mf(x) > \lambda\}.$$

Then $f = g + b$, *with*

$$\|g\|_2^2 \lesssim \lambda N (\log N)^2 \|f\|_1,$$

$$\left|\left\{x \notin E_\lambda : \sup_{j=1,\dots,N} \left|H(\mathrm{Mod}_{\xi_j} b)(x)\right| > \lambda\right\}\right| \lesssim N^{-100}\frac{\|f\|_1}{\lambda}.$$

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The bilinear Hilbert transform BHT

Family of trilinear forms indexed by unit vectors $\beta \cdot \vec{1} = 0$

$$\Lambda_\beta(f_1, f_2, f_3) := \int_{\xi_1+\xi_2+\xi_3=0} \mathrm{sign}(\beta \cdot \xi) \Big(\prod_{j=1}^{3} \widehat{f_j}(\xi_j) \Big) \, \mathrm{d}\sigma$$
$$= c \int_{\mathbb{R}} \Big(\prod_{j=1}^{3} f_j(x - \beta_j t) \Big) \, \mathrm{p.v.} \frac{\mathrm{d}t}{t}$$

dual to the bilinear Hilbert trasforms BHT_β

RMK. Note the one parameter modulation invariance

$$\Lambda_\beta(f_1, f_2, f_3) = \Lambda_\beta(\mathrm{Mod}_{\gamma_1} f_1, \mathrm{Mod}_{\gamma_2} f_2, \mathrm{Mod}_{\gamma_3} f_3)$$

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The boundedness region for Λ_β

Theorem (Lacey-Thiele97,99).

Away from $\beta_j = \beta_k$, Λ_β is of Generalized Restricted Weak Type

$$\forall \vec{\alpha} \in \mathcal{H} = \left\{ \alpha_1 + \alpha_2 + \alpha_3 = 1,\ \max \alpha_j < 1,\ \min \alpha_j > -\tfrac{1}{2} \right\}$$

GRWT interpolation implies

$$\mathrm{BHT}_\beta : L^{p_1} \times L^{p_2} \to L^q, \qquad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q},\ 1 < p_1, p_2 \le \infty,\ \frac{2}{3} < q < \infty.$$

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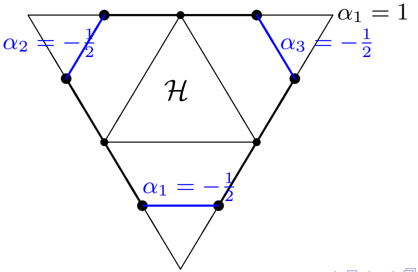

The $L^{\frac{2}{3}}$ threshold for BHT

The dual to BHT belongs to the family of modulation-invariant multipliers

$$\Lambda_{m_\beta}(f_1, f_2, f_3) := \int_{\xi_1+\xi_2+\xi_3=0} m_\beta(\xi) \Big(\prod_{j=1}^{3} \widehat{f}_j(\xi) \Big) \,\mathrm{d}\xi,$$

$$\big|\partial^\alpha m_\beta(\xi)\big| \lesssim \operatorname{dist}(\xi, \{t\gamma : t \in \mathbb{R}\})^{-|\alpha|}, \quad \{\vec{1}, \beta, \gamma\} \text{ ONB}$$

Counterexample. (Lacey-Thiele, Lacey, Muscalu)

- Discrete model for BHT_β does not map into L^q whenever $q < \frac{2}{3}$.
- For each β $\exists m_\beta$ as above s.t. T_{m_β} does not map into L^q, $q < \frac{2}{3}$

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An endpoint conjecture

THM (Log-failure of $\alpha_3 = -1/2$: Lacey-Thiele, Bilyk-Grafakos'06).

On the open segment $\alpha_3 = -\frac{1}{2}$, there holds the GRWT estimate (any three F_j, three restricted f_j)

$$\big|\Lambda_\beta(f_1, f_2, f_3)\big| \leq C_\beta |F_1|^{\alpha_1} |F_2|^{\alpha_2} |F_3|^{-\frac{1}{2}} \log\left(\mathrm{e} + \tfrac{|F_3|}{\min\{|F_1|,|F_2|\}}\right).$$

CONJECTURE 1. Λ_β is of GRWT α on the open segment $\alpha_3 = -\frac{1}{2} \equiv$

$$\mathrm{BHT}_\beta : L^{p_1,\frac{2}{3}} \times L^{p_2,\frac{2}{3}} \to L^{\frac{2}{3},\infty}, \qquad 1 < p_1, p_2 < 2,\ \frac{1}{p_1} + \frac{1}{p_2} = \frac{3}{2}$$

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Conjecture 1 up to a double logarithm

Theorem (DP-Thiele'14)

There holds the GRWT

$$\left|\Lambda_{\vec\beta}(f_1,f_2,f_3)\right| \lesssim_\beta |F_1|^{\frac34}|F_2|^{\frac34}|F_3|^{-\frac12}\left[\log\log\left(\mathrm{e}^{\mathrm{e}}+\frac{|F_3|}{\min\{|F_1|,|F_2|\}}\right)\right].$$

Equivalently

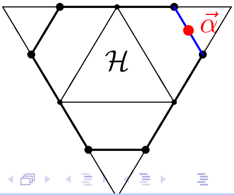
$$\mathrm{BHT}_\beta : L^{\frac43,\frac23}(\log\log L)^{\frac23}\times L^{\frac43,\frac23}(\log\log L)^{\frac23}\to L^{\frac23,\infty}(\log\log L)^{-1}$$

and similarly for other tuples $\vec\alpha$ with $\alpha_3=-\frac12$.

On $\alpha_3=-\frac12$, $f_1,f_2\in L^p$, $1<p<2$.

Proof: augmented multi-frequency decomposition

$\rightsquigarrow$ lower order correction terms



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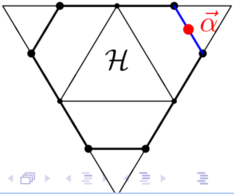

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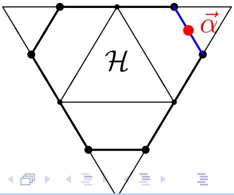

Banach-valued multilinear singular integrals

Long term goal: Banach space valued BHT. Setting

- Banach spaces $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$
- a trilinear form id : $\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3 \to \mathbb{C}$ such that

$$|\mathrm{id}(x_1, x_2, x_3)| \leq \prod_{j=1,2,3} \|x_j\|_{\mathcal{X}_j}, \qquad x_j \in \mathcal{X}_j,\ j = 1, 2, 3.$$

Examples:

- $\mathcal{X}_1 = \mathcal{X}, \mathcal{X}_2 = \mathbb{C}, \mathcal{X}_3 = \mathcal{X}'$, $\mathcal{X}$ UMD space.
- $\mathcal{X}_j$ three UMD function lattices on a measure space, id = Hölder ineq.

Aim. $L^p(\mathcal{X}_j)$ estimates for trilinear forms of type

$$\Lambda_m(f_1, f_2, f_3) = \int_{\xi_1+\xi_2+\xi_3=0} m(\xi_1, \xi_2, \xi_3)\left[\widehat{f_1}(\xi_1), \widehat{f_2}(\xi_2), \widehat{f_3}(\xi_3)\right] \mathrm{d}\xi,$$

with $m : \mathbb{R}^3 \to B(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3)$ *operator valued symbol.*

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An operator valued Coifman-Meyer theorem

Theorem (DP-Ou15)

- $\mathcal{X}_j$, $j = 1, 2, 3$ are UMD-RMF spaces
- $\mathcal{R}_{\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3}\left(\left\{|\xi|^{|\alpha|}\partial_\xi^\alpha m(\xi) : \xi \in \vec{1}^\perp\right\}\right) \leq C, \qquad |\alpha| \leq N$

Then the dual operator(s) T_m to Λ_m satisfy

- $T_m : L^{p_1}(\mathbb{R}; \mathcal{X}_1) \times L^{p_2}(\mathbb{R}; \mathcal{X}_2) \to L^{\frac{p_1 p_2}{p_1+p_2}}(\mathbb{R}; \mathcal{X}_3')$, $1 < p_1, p_2 < \infty$,
- $T_m : L^1(\mathbb{R}; \mathcal{X}_1) \times L^1(\mathbb{R}; \mathcal{X}_2) \to L^{\frac{1}{2},\infty}(\mathbb{R}; \mathcal{X}_3')$.

The condition on $\mathcal{R}_{\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3}$

- is the usual CM condition if $m = \tilde{m} \cdot \mathrm{id}$ for $\tilde{m}$ scalar multiplier;
- if $\mathcal{X}_j = L^{q_j}$ Hölder tuple, equivalent to

$$L^{q_1}(\ell^2) \times L^{q_2}(\ell^2) \times L^{q_3}(\ell^\infty)$$

bounds for sequence-valued operators with coordinates in collection

$$\mathcal{M} = \left\{|\xi|^{|\alpha|}\partial_\xi^\alpha m(\xi) : \xi_1 + \xi_2 + \xi_3 = 0, \ |\alpha| \leq N\right\}.$$

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Theorem (DP-Ou15)

- $\mathcal{X}_j$, $j = 1,2,3$ are UMD-RMF spaces
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Application: mixed norm estimates for $BHT \otimes PP$

Theorem (DP-Ou15, improving Silva; see also Benea-Muscalu).

Let $m = m(\xi, \eta)$ be defined on $\vec{1}^\perp \times \vec{1}^\perp$, satisfying

$$\sup_{\xi,\eta} |\xi|^{|\alpha|} \mathrm{dist}(\eta, \Gamma_1)^{|\beta|} \left|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)\right| \le K, \qquad |\alpha|, |\beta| \le N$$

where Γ_1 non-degenerate line in $\vec{1}^\perp$. Then, the dual operators T to

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Banach valued outer L^p space theory

Outer measure via tents, adapted from Do-Thiele

- $I \subset \mathbb{R} \leftrightarrow \mathsf{T}(I) = \{(y,t) \in \mathbb{R}^2_+ : 0 < t < \ell(I), |y-x| < \ell(I) - t\}$
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Let $F = F(y,t) : \mathbb{R}^2_+ \to \mathcal{X}$. Define $\Gamma_I(y) = \{(u,t) : |u-y| < t < \ell(I)\}$

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Carleson embedding theorems

For $f : \mathbb{R} \to \mathcal{X}$, $\lambda > 0$

$$F_\phi(f)(y,t) = \int f(z)\phi(t^{-1}(y-z))\frac{\mathrm{d}z}{t}, \qquad \widetilde{F_\phi}(f) := F_\phi(f)\mathbf{1}_{(E_\lambda)^c}$$

where $\widehat{E_\lambda}$ is the *tent over* $E_\lambda = \{Mf > \lambda\}$.

Hölder ineq. For $f_j : \mathbb{R} \to \mathcal{X}_j$, $j = 1,2,3$,

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Carleson embedding theorems: if ϕ mean zero and $\mathcal{X}$ UMD space

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$$\|F_\phi(f)\|_{L^\infty(S)} \lesssim \|f\|_{\mathrm{BMO}(\mathcal{X})}, \qquad \|F_\phi(f)\|_{L^p(S)} \lesssim \|f\|_{L^p(\mathcal{X})},\ 1 < p < \infty,$$

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where $\widehat{E_\lambda}$ is the *tent over* $E_\lambda = \{Mf > \lambda\}$.

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Outer measure and size via generalized tents on $\mathbb{R}_y \times \mathbb{R}_\eta \times (0,\infty)$

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