UNIVERSITEIT GENT

FACULTY OF **ENGINEERING AND ARCHITECTURE**

Slice Fourier transform: definition, properties and corresponding convolutions

L. Cnudde H. De Bie

Ghent University
Dept. of Mathematical Analysis
Clifford Research Group

June 10, 2015, Porto

Contents

Contents

UNIVERSITEIT GENT

What defines a Fourier transform?

- Its kernel expression?

$$\mathcal{F}(f)(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} f(x)\mathrm{d}x$$

- Its eigenvectors and eigenvalues?

$$\mathcal{F}(\psi_n) = (-i)^n \psi_n$$

- Its differential equations?

$$\partial_y \mathcal{F}(f) = -i\mathcal{F}(xf)$$
$$iy\mathcal{F}(f) = \mathcal{F}(\partial_x f)$$

- Its convolution property?

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$$

Classical Fourier transform in a nutshell

Definition

The classical Fourier transform is defined as

$$\mathcal{F} : L^2(\mathbb{R}, \mathrm{d}x) \to L^2(\mathbb{R}, \mathrm{d}x) : f(x) \mapsto \mathcal{F}(f)(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} f(x) \mathrm{d}x,$$

where

$$L^2(\mathbb{R}, \mathrm{d}x) = \left\{ f : \mathbb{R} \to \mathbb{R} \;\middle|\; \int_{-\infty}^{+\infty} (f(x))^* f(x) \, \mathrm{d}x < \infty \right\}.$$

Classical Fourier transform in a nutshell

Definition

The classical Fourier transform is defined as

$$\mathcal{F} : L^2(\mathbb{R}, \mathrm{d}x) \to L^2(\mathbb{R}, \mathrm{d}x) : f(x) \mapsto \mathcal{F}(f)(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} f(x) \mathrm{d}x,$$

where

$$L^2(\mathbb{R}, \mathrm{d}x) = \left\{ f : \mathbb{R} \to \mathbb{R} \;\middle|\; \int_{-\infty}^{+\infty} (f(x))^* f(x)\, \mathrm{d}x < \infty \right\}.$$

The L^2-space can be endowed with the standard inner product:

$$\langle f, g \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (f(x))^* g(x) \mathrm{d}x$$

Classical Fourier transform in a nutshell

Definition (Hermite polynomials)

The Hermite polynomial H_j of order j is defined as

$$H_j(x) = \left(2x - \frac{\mathrm{d}}{\mathrm{d}x}\right)^j 1.$$

One might think of 1 as a basis element of $\ker \frac{\mathrm{d}}{\mathrm{d}x}$.

Definition (Hermite functions)

The Hermite function ψ_j of order j is defined as

$$\psi_j(x) = H_j(x) \exp\left(\frac{-x^2}{2}\right).$$

With this definition and the above inner product, one has

$$\langle \psi_{j_1}, \psi_{j_2} \rangle = \langle \psi_{j_1}, \psi_{j_1} \rangle \delta_{j_1 j_2}.$$

Classical Fourier transform in a nutshell

Theorem (Laguerre formulation)

The Hermite polynomials H_j may be expressed as

$$H_{2t}(x) = (-1)^t 2^{2t} \quad t! \; L_t^{-1/2}\left(x^2\right)$$
$$H_{2t+1}(x) = (-1)^t 2^{2t+1} t! x L_t^{+1/2}\left(x^2\right).$$

where L_t^k are the generalised Laguerre polynomials of degree t.

Theorem (scalar differential equation)

One has

$$\left(x^2 - \frac{\mathrm{d}^2}{\mathrm{d}x^2} + 1\right)\psi_j(x) = 2(j+1)\psi_j(x)$$

Mehler approach

Therefore the following formal expression holds:

$$e^{-\frac{i\pi(H-1)}{4}}\psi_j(x) = (-i)^j \psi_j(x)$$

where the operator H is given by $H = x^2 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}$.

Mehler approach

Therefore the following formal expression holds:

$$e^{-\frac{i\pi(H-1)}{4}}\psi_j(x) = (-i)^j\psi_j(x)$$

where the operator H is given by $H = x^2 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}$.

Theorem (Eigenfunction equation)

$$\mathcal{F}(\psi_j) = (-i)^j\psi_j$$

Mehler approach

Therefore the following formal expression holds:

$$e^{-\frac{i\pi(H-1)}{4}}\psi_j(x) = (-i)^j\psi_j(x)$$

where the operator H is given by $H = x^2 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}$.

Theorem (Eigenfunction equation)

$$\mathcal{F}(\psi_j) = (-i)^j\psi_j$$

An integral expression can be constructed based on the orthogonality of the ψ_j's with respect to the inner product. With $f \in \mathrm{span}\{\psi_j\}$ one has

$$f(x)$$

Mehler approach

Therefore the following formal expression holds:

$$e^{-\frac{i\pi(H-1)}{4}}\psi_j(x) = (-i)^j\psi_j(x)$$

where the operator H is given by $H = x^2 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}$.

Theorem (Eigenfunction equation)

$$\mathcal{F}(\psi_j) = (-i)^j\psi_j$$

An integral expression can be constructed based on the orthogonality of the ψ_j's with respect to the inner product. With $f \in \mathrm{span}\{\psi_j\}$ one has

$$\int_{\mathbb{R}} \quad \frac{(\psi_j(x))^*}{\langle\psi_j,\psi_j\rangle} f(x)\,\mathrm{d}x$$

UNIVERSITEIT GENT

Mehler approach

Therefore the following formal expression holds:

$$e^{-\frac{i\pi(H-1)}{4}}\psi_j(x) = (-i)^j\psi_j(x)$$

where the operator H is given by $H = x^2 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}$.

Theorem (Eigenfunction equation)

$$\mathcal{F}(\psi_j) = (-i)^j\psi_j$$

An integral expression can be constructed based on the orthogonality of the ψ_j's with respect to the inner product. With $f \in \text{span}\{\psi_j\}$ one has

$$\int_{\mathbb{R}} \qquad \frac{(-i)^j(\psi_j(x))^*}{\langle\psi_j, \psi_j\rangle} f(x)\, \mathrm{d}x$$

Mehler approach

Therefore the following formal expression holds:

$$e^{-\frac{i\pi(H-1)}{4}}\psi_j(x) = (-i)^j\psi_j(x)$$

where the operator H is given by $H = x^2 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}$.

Theorem (Eigenfunction equation)

$$\mathcal{F}(\psi_j) = (-i)^j\psi_j$$

An integral expression can be constructed based on the orthogonality of the ψ_j's with respect to the inner product. With $f \in \text{span}\{\psi_j\}$ one has

$$\int_{\mathbb{R}} \qquad \frac{\psi_j(y)(-i)^j(\psi_j(x))^*}{\langle\psi_j,\psi_j\rangle} f(x)\,\mathrm{d}x$$

Mehler approach

Therefore the following formal expression holds:

$$e^{-\frac{i\pi(H-1)}{4}}\psi_j(x) = (-i)^j\psi_j(x)$$

where the operator H is given by $H = x^2 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}$.

Theorem (Eigenfunction equation)

$$\mathcal{F}(\psi_j) = (-i)^j\psi_j$$

An integral expression can be constructed based on the orthogonality of the ψ_j's with respect to the inner product. With $f \in \text{span}\{\psi_j\}$ one has

$$\mathcal{F}(f)(y) = \int_{\mathbb{R}} \sum_{j=0}^{\infty} \frac{\psi_j(y)(-i)^j(\psi_j(x))^*}{\langle\psi_j, \psi_j\rangle} f(x)\,\mathrm{d}x$$

Mehler approach

Therefore the following formal expression holds:

$$e^{-\frac{i\pi(H-1)}{4}}\psi_j(x) = (-i)^j\psi_j(x)$$

where the operator H is given by $H = x^2 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}$.

Theorem (Eigenfunction equation)

$$\mathcal{F}(\psi_j) = (-i)^j\psi_j$$

An integral expression can be constructed based on the orthogonality of the ψ_j's with respect to the inner product. With $f \in \mathrm{span}\{\psi_j\}$ one has

$$\mathcal{F}(f)(y) = \int_{\mathbb{R}} \sum_{j=0}^{\infty} \frac{\psi_j(y)(-i)^j(\psi_j(x))^*}{\langle\psi_j, \psi_j\rangle} f(x)\,\mathrm{d}x$$

$$= \int_{\mathbb{R}} \frac{e^{-ixy}}{\sqrt{2\pi}}\, f(x)\,\mathrm{d}x$$

UNIVERSITEIT GENT

Contents

UNIVERSITEIT GENT

Slice setting

Original framework

- Real Clifford algebra $\mathbb{R}^m$ with basis $\{\widetilde{e}_i, i = 1, \ldots, m\}$:

$$\widetilde{e}_i \widetilde{e}_j + \widetilde{e}_j \widetilde{e}_i = -2\delta_{ij}, \qquad i, j = 1, \ldots, m.$$

- Paravector $\widetilde{\mathrm{x}} = x_0 + r\underline{\widetilde{\omega}} \in \mathbb{R}^0_m \oplus \mathbb{R}^1_m$ with r the norm of $\underline{\widetilde{x}}$ and $\underline{\widetilde{\omega}} = \underline{\widetilde{x}}/r$ the unit vector along $\underline{\widetilde{x}}$.
- Associated Slice Cauchy-Riemann operator

$$\widetilde{D}_0^{CR} = \frac{\partial}{\partial x_0} + \underline{\widetilde{\omega}} \frac{\partial}{\partial r}$$

Slice setting

Original framework left-multiplied with e_0

- Real Clifford algebra $\mathbb{R}^m$ with basis $\{\widetilde{e}_i, i = 1, \ldots, m\}$:

$$\widetilde{e}_i\widetilde{e}_j + \widetilde{e}_j\widetilde{e}_i = -2\delta_{ij}, \qquad i, j = 1, \ldots, m.$$

- Paravector $\widetilde{x} = x_0 + r\underline{\widetilde{\omega}} \in \mathbb{R}^0_m \oplus \mathbb{R}^1_m$ with r the norm of $\underline{\widetilde{x}}$ and $\underline{\widetilde{\omega}} = \underline{\widetilde{x}}/r$ the unit vector along $\underline{\widetilde{x}}$.
- Associated Slice Cauchy-Riemann operator

$$\widetilde{D}_0^{CR} = \frac{\partial}{\partial x_0} + \underline{\widetilde{\omega}}\frac{\partial}{\partial r}$$

Slice setting

Original framework left-multiplied with e_0

- Real Clifford algebra $\mathbb{R}^{m+1}$ with basis $\{e_i, i = 0, \ldots, m\}$ where $e_i = e_0 \widetilde{e}_i, i = 1, \ldots, m$:

$$e_i e_j + e_j e_i = -2\delta_{ij}, \qquad i, j = 0, \ldots, m.$$

- 1-vector $\mathbf{x} = x_0 e_0 + r\underline{\omega} \in \mathbb{R}^1_{m+1}$ with r the norm of $\underline{x}$ and $\underline{\omega} = \underline{x}/r$ the unit vector along $\underline{x}$.
- Associated slice Dirac operator

$$D_0 = e_0 \frac{\partial}{\partial x_0} + \underline{\omega} \frac{\partial}{\partial r}$$

Slice setting

Theorem ($\mathfrak{osp}(1|2)$-superalgebra)

Defining the Euler operator $\mathbb{E}$ as

$$\mathbb{E} = \sum_{i=0}^{m} x_i \frac{\partial}{\partial x_i},$$

the operators $\mathbf{x}$, D_0 and $\mathbb{E}$ constitute a Lie superalgebra isomorphic with $\mathfrak{osp}(1|2)$, with relations

$$\begin{array}{ll}
\{\mathbf{x}, \mathbf{x}\} = -2|\mathbf{x}|^2 & \{D_0, D_0\} = -2(\partial_{x_0}^2 + \partial_r^2) \\
\{\mathbf{x}, D_0\} = -2\left(\mathbb{E} + 1\right) & [\mathbb{E} + 1, D_0] = -D_0 \\
[|\mathbf{x}|^2, D_0] = -2\mathbf{x} & [\mathbb{E} + 1, \mathbf{x}] = \mathbf{x} \\
[\partial_{x_0}^2 + \partial_r^2, \mathbf{x}] = 2D_0 & [\mathbb{E} + 1, \partial_{x_0}^2 + \partial_r^2] = -2(\partial_{x_0}^2 + \partial_r^2) \\
[\partial_{x_0}^2 + \partial_r^2, |\mathbf{x}|^2] = 4\left(\mathbb{E} + 1\right) & [\mathbb{E} + 1, |\mathbf{x}|^2] = 2|\mathbf{x}|^2.
\end{array}$$

Eigenfunctions and eigenvalues

The space of k-homogeneous polynomials ($k \in \mathbb{N}$) in the kernel of D_0 is one-dimensional and a basis is given by

$$m_k(\mathbf{x}) = (e_0 - 1)\,(x_0 + \underline{x})^k \mathbf{a}, \qquad \mathbf{a} \in Cl_{m+1}.$$

Definition (Hermite polynomials)

The Hermite polynomials $H_j(m_k)$ for the slice Dirac operator are defined as

$$H_j(m_k)(\mathbf{x}) = (\mathbf{x} - cD_0)^j\, m_k(\mathbf{x})$$

with $c \in \mathbb{R}_0^+$ a strictly positive, real parameter.

Definition

The Clifford-Hermite functions $\psi_{j,k}$ are defined as

$$\psi_{j,k}(\mathbf{x}) = H_j(m_k)(\mathbf{x}) \exp\left(-\frac{|\mathbf{x}|^2}{4c}\right).$$

Eigenfunctions and eigenvalues

Theorem (Laguerre formulation)

The Hermite polynomials $H_j(m_k)$ may be expressed as

$$H_{2t}(m_k)(\mathbf{x}) = (2c)^t t!\ L_t^k \left(\frac{|\mathbf{x}|^2}{2c} \right) m_k(\mathbf{x})$$

$$H_{2t+1}(m_k)(\mathbf{x}) = (2c)^t t!\ \mathbf{x}\ L_t^{k+1} \left(\frac{|\mathbf{x}|^2}{2c} \right) m_k(\mathbf{x})$$

where L_t^k are the generalised Laguerre polynomials of degree t.

Theorem (scalar differential equation)

The Clifford-Hermite functions $\psi_{j,k}$ are eigenfunctions of the scalar differential equation

$$\left(cD_0^2 + \frac{|\mathbf{x}|^2}{4c} \right) \psi_{j,k}(\mathbf{x}) = (j + k + 1)\psi_{j,k}(\mathbf{x}).$$

L. Cnudde, H. De Bie, and G. Ren.
Algebraic approach to slice monogenic functions.
Complex Analysis and Operator Theory **9** (2015), 1065–1087.

Mehler construction

- Because of the scalar differential equation, its formal expression is

$$e^{-i\frac{\pi}{2}H}\psi_{j,k}(\mathbf{x}) = (-i)^{(j+k+1)}\psi_{j,k}(\mathbf{x})$$

- Construct slice Fourier transform such that Clifford-Hermite functions $\psi_{j,k}$ are eigenfunctions with corresponding eigenvalues.
- For an integral expression an inner product is needed such that the $\psi_{j,k}$'s are orthogonal.

Mehler construction

- Because of the scalar differential equation, its formal expression is

$$e^{-i\frac{\pi}{2}H}\psi_{j,k}(\mathbf{x}) = (-i)^{(j+k+1)}\psi_{j,k}(\mathbf{x})$$

- Construct slice Fourier transform such that Clifford-Hermite functions $\psi_{j,k}$ are eigenfunctions with corresponding eigenvalues.
- For an integral expression an inner product is needed such that the $\psi_{j,k}$'s are orthogonal.

Definition (Inner product)

$$\langle f, g\rangle = \left[\int_{\mathbb{R}^{m+1}} \overline{f}\ g\ r^{1-m}\ \mathrm{d}\mathbf{x}\right]_0 = \left[\int_{\mathbb{R}^{m+1}} \overline{f}\ g\ \mathrm{d}x_0 \mathrm{d}r \mathrm{d}\sigma_{\mathbf{x}}\right]_0 \text{ where}$$

$$f, g \in \mathcal{L}_2 = \left\{ f : \mathbb{R}^{m+1} \to Cl_{m+1} \;\middle|\; \left[\int_{\mathbb{R}^{m+1}} \overline{f(\mathbf{x})} f(\mathbf{x})\ r^{1-m}\ \mathrm{d}\mathbf{x}\right]_0 < +\infty \right\}.$$

Mehler construction

Proposition

On a dense subset of $\mathcal{L}_2$, the inner product shows the relations

$$\langle D_0 f, g\rangle = \langle f, D_0 g\rangle,$$
$$\langle \mathbf{x} f, g\rangle = -\langle f, \mathbf{x} g\rangle.$$

Because $\begin{cases} \psi_{j,k} &= \left(\frac{\mathbf{x}}{2} - cD_0\right)\psi_{j-1,k} \\ \left(\frac{\mathbf{x}}{2} + cD_0\right)\psi_{j,k} &= c\ C(j,k)\psi_{j-1,k} \end{cases}$, we get

Mehler construction

Proposition

On a dense subset of $\mathcal{L}_2$, the inner product shows the relations

$$\langle D_0 f, g\rangle = \langle f, D_0 g\rangle,$$
$$\langle \mathbf{x} f, g\rangle = -\langle f, \mathbf{x} g\rangle.$$

Because $\begin{cases} \psi_{j,k} & = \left(\frac{\mathbf{x}}{2} - cD_0\right)\psi_{j-1,k} \\ \left(\frac{\mathbf{x}}{2} + cD_0\right)\psi_{j,k} & = c\ C(j,k)\psi_{j-1,k} \end{cases}$, we get

Theorem (Orthogonality theorem)

The inner product of two Clifford-Hermite functions ψ_{j_1,k_1} and ψ_{j_2,k_2} reads

$$\langle \psi_{j_1,k_1}, \psi_{j_2,k_2}\rangle = A(j_1,k_1)\delta_{j_1 j_2}\delta_{k_1 k_2}$$

where $A(j_1,k_1) \in \mathbb{R}_0^+$.

Mehler construction

The formal expression for the kernel function is given by

$$\mathcal{K}^M(\mathbf{x},\mathbf{y}) = \sum_{j,k=0}^{+\infty} \frac{\psi_{j,k}(\mathbf{y})(-i)^{j+k+1}\overline{\psi_{j,k}(\mathbf{x})}}{\langle \psi_{j,k}, \psi_{j,k} \rangle}.$$

Mehler construction

The formal expression for the kernel function is given by

$$\mathcal{K}^M(\mathbf{x}, \mathbf{y}) = \sum_{j,k=0}^{+\infty} \frac{\psi_{j,k}(\mathbf{y})(-i)^{j+k+1}\overline{\psi_{j,k}(\mathbf{x})}}{\langle \psi_{j,k}, \psi_{j,k} \rangle}.$$

Using the definitions and properties of the Clifford-Hermite functions $\psi_{j,k}$ this expression can be simplified by

- substituting the known expressions for $\psi_{j,k}$ and $\langle \psi_{j,k}, \psi_{j,k} \rangle$
- using the Hille-Hardy formula
- using trigonometric identities
- writing a Bessel series as an exponential

Mehler construction

The Mehler kernel function $\mathcal{K}^M$ thus becomes

$$\mathcal{K}^M(\mathbf{x},\mathbf{y}) \sim \Bigg[2J_0\left(-\frac{|\mathbf{x}||\mathbf{y}|}{2c}\right)$$
$$+(1-i) \quad \left(\sum_{k=1}^{+\infty}\frac{(y_0+\underline{y})^k(x_0-\underline{x})^k+(y_0-\underline{y})^k(x_0+\underline{x})^k}{(|\mathbf{x}||\mathbf{y}|)^k}(-i)^k J_k\left(-\frac{|\mathbf{x}||\mathbf{y}|}{2c}\right)\right)$$
$$+(1+i)e_0\left(\sum_{k=1}^{+\infty}\frac{(y_0+\underline{y})^k(x_0-\underline{x})^k-(y_0-\underline{y})^k(x_0+\underline{x})^k}{(|\mathbf{x}||\mathbf{y}|)^k}(-i)^k J_k\left(-\frac{|\mathbf{x}||\mathbf{y}|}{2c}\right)\right)\Bigg].$$

UNIVERSITEIT GENT

Mehler construction

The Mehler kernel function $\mathcal{K}^M$ thus becomes

$$\mathcal{K}^M(\mathbf{x},\mathbf{y}) \sim \Bigg[2J_0\left(-\frac{|\mathbf{x}||\mathbf{y}|}{2c}\right)$$
$$+(1-i) \quad \left(\sum_{k=1}^{+\infty}\frac{(y_0+\underline{y})^k(x_0-\underline{x})^k+(y_0-\underline{y})^k(x_0+\underline{x})^k}{(|\mathbf{x}||\mathbf{y}|)^k}(-i)^k J_k\left(-\frac{|\mathbf{x}||\mathbf{y}|}{2c}\right)\right)$$
$$+(1+i)e_0\left(\sum_{k=1}^{+\infty}\frac{(y_0+\underline{y})^k(x_0-\underline{x})^k-(y_0-\underline{y})^k(x_0+\underline{x})^k}{(|\mathbf{x}||\mathbf{y}|)^k}(-i)^k J_k\left(-\frac{|\mathbf{x}||\mathbf{y}|}{2c}\right)\right)\Bigg].$$

Writing

$$\left(\frac{(y_0+ig)}{\sqrt{y_0^2+g^2}}\right)^k = \cos(k\phi)+i\sin(k\phi) \text{ and } \left(\frac{(x_0+ir)}{\sqrt{x_0^2+r^2}}\right)^k = \cos(k\chi)+i\sin(k\chi)$$

we get

$$\mathcal{K}^M(\mathbf{x},\mathbf{y}) \sim \Bigg[J_0+\left(\sum_{k=1}^{+\infty}\left[\cos(k(\phi+\chi))+\cos(k(\phi-\chi))\right](-i)^k J_k\right)$$
$$-\left(\sum_{k=1}^{+\infty}\left[\cos(k(\phi-\chi))-\cos(k(\phi+\chi))\right](-i)^k J_k\right)\underline{\eta}\underline{\omega}\Bigg].$$

UNIVERSITEIT GENT

Mehler construction

Finally, using the identity

$$e^{iz\cos(\phi)} = J_0(z) + 2\sum_{n=1}^{+\infty} i^n J_n(z)\cos(n\phi),$$

the Mehler kernel $\mathcal{K}^M$ is given by

$$\mathcal{K}^M(\mathbf{x},\mathbf{y}) = \frac{-i\Gamma(m/2)}{8c\pi^{m/2+1}}\left[(1+\underline{\eta}\underline{\omega})e^{\frac{-i}{2c}(x_0y_0-rg)} + (1-\underline{\eta}\underline{\omega})e^{\frac{-i}{2c}(x_0y_0+rg)}\right].$$

Definition (slice Fourier transform)

The slice Fourier transform $\mathcal{F}_S$ is defined as

$$\mathcal{F}_S(f)(\mathbf{y}) = \int_{\mathbb{R}^{m+1}} \mathcal{K}^M(\mathbf{x},\mathbf{y})\ f(\mathbf{x})\ \mathrm{d}x_0\mathrm{d}r\mathrm{d}\sigma_{\mathbf{x}}.$$

Properties

For $f \in \text{span}\{\psi_{j,k}\}$, one has

- Denoting a translation in the e_0-direction over a distance a as $t_a f(x_0, r, \underline{\omega}) = f(x_0 - a, r, \underline{\omega})$, one has

$$\mathcal{F}_S(t_a f)(\mathbf{y}) = e^{-\frac{iay_0}{2c}} \mathcal{F}_S(f)(\mathbf{y}).$$

- Denoting a reflection with respect to the origin as $sf(x_0, r, \underline{\omega}) = f(-x_0, r, -\underline{\omega})$, one has

$$\mathcal{F}_S(sf)(\mathbf{y}) = s\mathcal{F}_S(f)(\mathbf{y})$$

- Denoting the complex conjugate of f as f^*, one has

$$\mathcal{F}_S(f^*)(\mathbf{y}) = -\left(\mathcal{F}_S(f)(-y_0, g, -\underline{\eta})\right)^*$$

- With respect to e_0, one has $\mathcal{F}_S(e_0 f)(\mathbf{y}) = e_0 \mathcal{F}_S(f)(\mathbf{y})$
- The twofold slice Fourier transform gives $\mathcal{F}_S(\mathcal{F}_S(f))(\mathbf{y}) = -f(-\mathbf{y})$

Properties

$$\mathcal{K}^M(\mathbf{x}, \mathbf{y}) = \sum_{j,k=0}^{+\infty} \frac{\psi_{j,k}(\mathbf{y})(-i)^{j+k+1}\overline{\psi_{j,k}(\mathbf{x})}}{\langle \psi_{j,k}, \psi_{j,k} \rangle},$$

Properties

- The following system of differential equations holds

$$\begin{cases} \frac{-i}{2c}\mathbf{y}\mathcal{K}^M(\mathbf{x},\mathbf{y}) & = \left[\mathcal{K}^M(\mathbf{x},\mathbf{y})D_0^{\mathbf{x}}\right] \\ D_0^{\mathbf{y}}\mathcal{K}^M(\mathbf{x},\mathbf{y}) & = \frac{i}{2c}\mathcal{K}^M(\mathbf{x},\mathbf{y})\mathbf{x} \end{cases}.$$

$$\mathcal{K}^M(\mathbf{x},\mathbf{y}) = \sum_{j,k=0}^{+\infty} \frac{\psi_{j,k}(\mathbf{y})(-i)^{j+k+1}\overline{\psi_{j,k}(\mathbf{x})}}{\langle \psi_{j,k}, \psi_{j,k}\rangle},$$

Properties

- The following system of differential equations holds

$$\begin{cases} \frac{-i}{2c}\mathbf{y}\mathcal{K}^M(\mathbf{x},\mathbf{y}) & = & \left[\mathcal{K}^M(\mathbf{x},\mathbf{y})D_0^{\mathbf{x}}\right] \\ D_0^{\mathbf{y}}\mathcal{K}^M(\mathbf{x},\mathbf{y}) & = & \frac{i}{2c}\mathcal{K}^M(\mathbf{x},\mathbf{y})\mathbf{x} \end{cases}.$$

- The kernel for the inverse transform $\mathcal{K}_{-1}^M$ is given by

$$\begin{aligned} \mathcal{K}_{-1}^M(\mathbf{y},\mathbf{x}) &= \frac{i\Gamma(m/2)}{8c\pi^{m/2+1}}\left[(1+\underline{\omega}\underline{\eta})e^{\frac{i}{2c}(x_0y_0-rg)} + (1-\underline{\omega}\underline{\eta})e^{\frac{i}{2c}(x_0y_0+rg)}\right] \\ &= \left(\mathcal{K}^M(\mathbf{y},\mathbf{x})\right)^*. \end{aligned}$$

UNIVERSITEIT GENT

Properties

- The following system of differential equations holds

$$\begin{cases} \frac{-i}{2c}\mathbf{y}\mathcal{K}^M(\mathbf{x},\mathbf{y}) &= \left[\mathcal{K}^M(\mathbf{x},\mathbf{y})D_0^{\mathbf{x}}\right] \\ D_0^{\mathbf{y}}\mathcal{K}^M(\mathbf{x},\mathbf{y}) &= \frac{i}{2c}\mathcal{K}^M(\mathbf{x},\mathbf{y})\mathbf{x} \end{cases}.$$

- The kernel for the inverse transform $\mathcal{K}_{-1}^M$ is given by

$$\begin{aligned} \mathcal{K}_{-1}^M(\mathbf{y},\mathbf{x}) &= \frac{i\Gamma(m/2)}{8c\pi^{m/2+1}}\left[(1+\underline{\omega}\underline{\eta})e^{\frac{i}{2c}(x_0y_0-rg)} + (1-\underline{\omega}\underline{\eta})e^{\frac{i}{2c}(x_0y_0+rg)}\right] \\ &= \left(\mathcal{K}^M(\mathbf{y},\mathbf{x})\right)^*. \end{aligned}$$

- The computational load of the slice Fourier transform equals that of two classical Fourier transforms.

Properties (computational load)

Indeed, given that

- $f \in \text{span}\{\psi_{j,k}\}$ so

$$f(\mathbf{x}) = f_0(x_0, r) + e_0 f_1(x_0, r) + \underline{\omega} f_2(x_0, r) + e_0 \underline{\omega} f_3(x_0, r),$$

- the classical two dimensional Fourier transform is proportional to

$$\int_{\mathbb{R}^2} \cos(xy)\cos(rg) f(x,r)\mathrm{d}x\mathrm{d}r - \int_{\mathbb{R}^2} \sin(xy)\sin(rg) f(x,r)\mathrm{d}x\mathrm{d}r$$
$$- i\int_{\mathbb{R}^2} \sin(xy)\cos(rg) f(x,r)\mathrm{d}x\mathrm{d}r - i\int_{\mathbb{R}^2} \cos(xy)\sin(rg) f(x,r)\mathrm{d}x\mathrm{d}r,$$

- a general function $f(x, r)$ can be written as

$$f(x,r) = f^{++}(x,r) + f^{+-}(x,r) + f^{-+}(x,r) + f^{--}(x,r),$$

the slice Fourier transform of $f \in \text{span}\{\psi_{j,k}\}$ equals

UNIVERSITEIT GENT

Properties (computational load)

$$
\begin{aligned}
&\mathcal{F}_S(f)(y) = \\
&\frac{-2i}{4\pi c}\Bigg(\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\cos\left(\frac{x_0 y_0}{2c}\right)\cos\left(\frac{rg}{2c}\right) f_0(x_0, r)\mathrm{d}x_0\mathrm{d}r + e_0\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\cos\left(\frac{x_0 y_0}{2c}\right)\cos\left(\frac{rg}{2c}\right) f_1(x_0, r)\mathrm{d}x_0\mathrm{d}r \\
&- \underline{\eta}\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\sin\left(\frac{x_0 y_0}{2c}\right)\sin\left(\frac{rg}{2c}\right) f_2(x_0, r)\mathrm{d}x_0\mathrm{d}r - e_0\underline{\eta}\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\sin\left(\frac{x_0 y_0}{2c}\right)\sin\left(\frac{rg}{2c}\right) f_3(x_0, r)\mathrm{d}x_0\mathrm{d}r \\
&- i\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\sin\left(\frac{x_0 y_0}{2c}\right)\cos\left(\frac{rg}{2c}\right) f_0(x_0, r)\mathrm{d}x_0\mathrm{d}r - ie_0\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\sin\left(\frac{x_0 y_0}{2c}\right)\cos\left(\frac{rg}{2c}\right) f_1(x_0, r)\mathrm{d}x_0\mathrm{d}r \\
&- i\underline{\eta}\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\cos\left(\frac{x_0 y_0}{2c}\right)\sin\left(\frac{rg}{2c}\right) f_2(x_0, r)\mathrm{d}x_0\mathrm{d}r - ie_0\underline{\eta}\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\cos\left(\frac{x_0 y_0}{2c}\right)\sin\left(\frac{rg}{2c}\right) f_3(x_0, r)\mathrm{d}x_0\mathrm{d}r\Bigg)
\end{aligned}
$$

Properties (computational load)

$$
\begin{aligned}
&\mathcal{F}_S(f)(y) = \\
&\frac{-2i}{4\pi c}\Bigg(\int_{-\infty}^{+\infty}\int_0^{+\infty}\cos\left(\frac{x_0 y_0}{2c}\right)\cos\left(\frac{rg}{2c}\right) f_0^+(x_0, r)\mathrm{d}x_0\mathrm{d}r + e_0\int_{-\infty}^{+\infty}\int_0^{+\infty}\cos\left(\frac{x_0 y_0}{2c}\right)\cos\left(\frac{rg}{2c}\right) f_1^+(x_0, r)\mathrm{d}x_0\mathrm{d}r \\
&-\underline{\eta}\int_{-\infty}^{+\infty}\int_0^{+\infty}\sin\left(\frac{x_0 y_0}{2c}\right)\sin\left(\frac{rg}{2c}\right) f_2^-(x_0, r)\mathrm{d}x_0\mathrm{d}r - e_0\underline{\eta}\int_{-\infty}^{+\infty}\int_0^{+\infty}\sin\left(\frac{x_0 y_0}{2c}\right)\sin\left(\frac{rg}{2c}\right) f_3^-(x_0, r)\mathrm{d}x_0\mathrm{d}r \\
&-i\int_{-\infty}^{+\infty}\int_0^{+\infty}\sin\left(\frac{x_0 y_0}{2c}\right)\cos\left(\frac{rg}{2c}\right) f_0^-(x_0, r)\mathrm{d}x_0\mathrm{d}r - ie_0\int_{-\infty}^{+\infty}\int_0^{+\infty}\sin\left(\frac{x_0 y_0}{2c}\right)\cos\left(\frac{rg}{2c}\right) f_1^-(x_0, r)\mathrm{d}x_0\mathrm{d}r \\
&-i\underline{\eta}\int_{-\infty}^{+\infty}\int_0^{+\infty}\cos\left(\frac{x_0 y_0}{2c}\right)\sin\left(\frac{rg}{2c}\right) f_2^+(x_0, r)\mathrm{d}x_0\mathrm{d}r - ie_0\underline{\eta}\int_{-\infty}^{+\infty}\int_0^{+\infty}\cos\left(\frac{x_0 y_0}{2c}\right)\sin\left(\frac{rg}{2c}\right) f_3^+(x_0, r)\mathrm{d}x_0\mathrm{d}r\Bigg)
\end{aligned}
$$

UNIVERSITEIT GENT

Properties (computational load)

$$\mathcal{F}_S(f)(y) =$$

$$\frac{-2i}{4\pi c}\left(\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\cos\left(\frac{x_0 y_0}{2c}\right)\cos\left(\frac{rg}{2c}\right) f_0^{++}(x_0, r)\mathrm{d}x_0\,\mathrm{d}r + e_0\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\cos\left(\frac{x_0 y_0}{2c}\right)\cos\left(\frac{rg}{2c}\right) f_1^{++}(x_0, r)\mathrm{d}x_0\,\mathrm{d}r\right.$$

$$-\underline{\eta}\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\sin\left(\frac{x_0 y_0}{2c}\right)\sin\left(\frac{rg}{2c}\right) f_2^{--}(x_0, r)\mathrm{d}x_0\,\mathrm{d}r - e_0\underline{\eta}\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\sin\left(\frac{x_0 y_0}{2c}\right)\sin\left(\frac{rg}{2c}\right) f_3^{--}(x_0, r)\mathrm{d}x_0\,\mathrm{d}r$$

$$-i\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\sin\left(\frac{x_0 y_0}{2c}\right)\cos\left(\frac{rg}{2c}\right) f_0^{-+}(x_0, r)\mathrm{d}x_0\,\mathrm{d}r - ie_0\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\sin\left(\frac{x_0 y_0}{2c}\right)\cos\left(\frac{rg}{2c}\right) f_1^{-+}(x_0, r)\mathrm{d}x_0\,\mathrm{d}r$$

$$\left.-i\underline{\eta}\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\cos\left(\frac{x_0 y_0}{2c}\right)\sin\left(\frac{rg}{2c}\right) f_2^{+-}(x_0, r)\mathrm{d}x_0\,\mathrm{d}r - ie_0\underline{\eta}\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\cos\left(\frac{x_0 y_0}{2c}\right)\sin\left(\frac{rg}{2c}\right) f_3^{+-}(x_0, r)\mathrm{d}x_0\,\mathrm{d}r\right)$$

UNIVERSITEIT GENT

Properties (computational load)

$$\mathcal{F}_S(f)(y) =$$

$$\frac{-i}{4\pi c}\Bigg(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cos\left(\frac{x_0y_0}{2c}\right)\cos\left(\frac{rg}{2c}\right)f_0^{++}(x_0,r)\mathrm{d}x_0\mathrm{d}r + e_0\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cos\left(\frac{x_0y_0}{2c}\right)\cos\left(\frac{rg}{2c}\right)f_1^{++}(x_0,r)\mathrm{d}x_0\mathrm{d}r$$

$$-\underline{\eta}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\sin\left(\frac{x_0y_0}{2c}\right)\sin\left(\frac{rg}{2c}\right)f_2^{--}(x_0,r)\mathrm{d}x_0\mathrm{d}r - e_0\underline{\eta}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\sin\left(\frac{x_0y_0}{2c}\right)\sin\left(\frac{rg}{2c}\right)f_3^{--}(x_0,r)\mathrm{d}x_0\mathrm{d}r$$

$$-i\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\sin\left(\frac{x_0y_0}{2c}\right)\cos\left(\frac{rg}{2c}\right)f_0^{-+}(x_0,r)\mathrm{d}x_0\mathrm{d}r - ie_0\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\sin\left(\frac{x_0y_0}{2c}\right)\cos\left(\frac{rg}{2c}\right)f_1^{-+}(x_0,r)\mathrm{d}x_0\mathrm{d}r$$

$$-i\underline{\eta}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cos\left(\frac{x_0y_0}{2c}\right)\sin\left(\frac{rg}{2c}\right)f_2^{+-}(x_0,r)\mathrm{d}x_0\mathrm{d}r - ie_0\underline{\eta}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cos\left(\frac{x_0y_0}{2c}\right)\sin\left(\frac{rg}{2c}\right)f_3^{+-}(x_0,r)\mathrm{d}x_0\mathrm{d}r\Bigg)$$

and therefore

$$\mathcal{F}_S(f) = \frac{-i}{2c}\left(2DFT(f_0^+ + \underline{\eta}f_2^-) + e_0\ 2DFT(f_1^+ + \underline{\eta}f_3^-)\right)$$

Contents

UNIVERSITEIT GENT

Convolutions

There are multiple ways to generalise the classical convolution defined as

$$f \star g = \int_{-\infty}^{+\infty} f(x-t)g(t)\mathrm{d}t = \int_{-\infty}^{+\infty} \tau_t f(x)g(t)\mathrm{d}t.$$

Again referring to the classical Fourier transform, one can choose to...

R. Bujack, H. De Bie, N. De Schepper and G. Scheuermann,
Convolution products for hypercomplex Fourier transforms,
J. Math Imaging Vis, volume 48 (2014).

Convolutions

There are multiple ways to generalise the classical convolution defined as

$$f \star g = \int_{-\infty}^{+\infty} f(x-t)g(t)\mathrm{d}t = \int_{-\infty}^{+\infty} \tau_t f(x)g(t)\mathrm{d}t.$$

Again referring to the classical Fourier transform, one can choose to...

- focus on the well-known property

$$\mathcal{F}(f \star g) = \mathcal{F}(f)\mathcal{F}(g)$$

- focus on the expression for the translation τ_t for which:

$$\mathcal{F}(\tau_t f)(y) = e^{-ity}\mathcal{F}(f)(y)$$

so

$$\tau_t f(x) = \mathcal{F}^{-1}\left(e^{-ity}\mathcal{F}(f)\right)(x)$$

R. Bujack, H. De Bie, N. De Schepper and G. Scheuermann,
Convolution products for hypercomplex Fourier transforms,
J. Math Imaging Vis, volume 48 (2014).

Mustard convolutions

Defined as

$$f \star_M^L g = \mathcal{F}_S^{-1}\left(\mathcal{F}_S(f)\mathcal{F}_S(g)\right),$$

the expression for the left Mustard convolution can be calculated as

$$(f \star_M^L g)(\mathbf{x}) = i\left(\frac{\Gamma\left(\frac{m}{2}\right)}{8c\pi^{m/2+1}}\right)^3 \int_{\mathbf{z}}\int_{\mathbf{u}}\int_{\mathbf{y}} \left[(1+\underline{\omega}\underline{\eta})e^{-\frac{i}{2c}(x_0y_0-rg)} + (1-\underline{\omega}\underline{\eta})e^{-\frac{i}{2c}(x_0y_0+rg)}\right]$$
$$\left[(1+\underline{\eta}\zeta)e^{-\frac{i}{2c}(z_0y_0-ng)} + (1-\underline{\eta}\zeta)e^{-\frac{i}{2c}(z_0y_0+ng)}\right] f(\mathbf{z})$$
$$\left[(1+\underline{\eta}\chi)e^{-\frac{i}{2c}(u_0y_0-mg)} + (1-\underline{\eta}\chi)e^{-\frac{i}{2c}(u_0y_0+mg)}\right] g(\mathbf{u})$$
$$\mathrm{d}z_0\ \mathrm{d}n\ \mathrm{d}\sigma_{\mathbf{z}}\ \mathrm{d}u_0\ \mathrm{d}m\ \mathrm{d}\sigma_{\mathbf{u}}\ \mathrm{d}y\ \mathrm{d}g\ \mathrm{d}\sigma_{\mathbf{y}}$$

UNIVERSITEIT GENT

Mustard convolutions

Defined as

$$f \star_M^L g = \mathcal{F}_S^{-1}\left(\mathcal{F}_S(f)\mathcal{F}_S(g)\right),$$

the expression for the left Mustard convolution can be calculated as

$$
\begin{aligned}
(f \star_M^L g)(\mathbf{x}) = i \left(\frac{\Gamma\left(\frac{m}{2}\right)}{8c\pi^{m/2+1}} \right)^3 \int_{\mathbf{z}} \int_{\mathbf{u}} \int_{\mathbf{y}} & \left[(1+\underline{\omega}\underline{\eta}) e^{-\frac{i}{2c}(x_0 y_0 - rg)} + (1-\underline{\omega}\underline{\eta}) e^{-\frac{i}{2c}(x_0 y_0 + rg)} \right] \\
& \left[(1+\underline{\eta}\zeta) e^{-\frac{i}{2c}(z_0 y_0 - ng)} + (1-\underline{\eta}\zeta) e^{-\frac{i}{2c}(z_0 y_0 + ng)} \right] f(\mathbf{z}) \\
& \left[(1+\underline{\eta}\chi) e^{-\frac{i}{2c}(u_0 y_0 - mg)} + (1-\underline{\eta}\chi) e^{-\frac{i}{2c}(u_0 y_0 + mg)} \right] g(\mathbf{u}) \\
& \mathrm{d}z_0 \, \mathrm{d}n \, \mathrm{d}\sigma_{\mathbf{z}} \, \mathrm{d}u_0 \, \mathrm{d}m \, \mathrm{d}\sigma_{\mathbf{u}} \, \mathrm{d}y \, \mathrm{d}g \, \mathrm{d}\sigma_{\mathbf{y}}
\end{aligned}
$$

Theorem

With $\underline{a} \in Cl_m$, one has

$$\int_{\mathbb{S}^{m-1}} \underline{\omega} \; \underline{a} \; \underline{\omega} \; \mathrm{d}\underline{\omega} = \frac{2\pi^{m/2}}{m\Gamma\left(\frac{m}{2}\right)} \sum_{k=0}^{m} (-1)^k (2k-m) \underline{a}^{(k)}.$$

Mustard convolutions

Writing $f(\mathbf{z}) = f_1(z_0, n) + \underline{\zeta} f_2(z_0, n)$, $f \in span\{\psi_{j,k}\}$, we get

$$\mathcal{F}^{-1}\left(\mathcal{F}(f)\mathcal{F}(g)\right)(\mathbf{x}) = \frac{-i}{4\pi c} \int_{-\infty}^{+\infty}$$

$$\left\{ \int_0^r \left[f_1(z_0, n) g_1(x_0 - z_0, r - n) + \sum_{k=0}^{m} (-1)^k \left(\frac{2k}{m} - 1\right) f_2^{(k)}(z_0, n) g_2(x_0 - z_0, r - n) \right] \mathrm{d}n \right.$$

$$+ \int_r^{+\infty} \left[f_1(z_0, n) g_1(x_0 - z_0, n - r) - \sum_{k=0}^{m} (-1)^k \left(\frac{2k}{m} - 1\right) f_2^{(k)}(z_0, n) g_2(x_0 - z_0, n - r) \right] \mathrm{d}n$$

$$\left. + \int_0^{+\infty} \left[f(z_0, n) g_1(x_0 - z_0, r + n) - \sum_{k=0}^{m} (-1)^k \left(\frac{2k}{m} - 1\right) f_2^{(k)}(z_0, n) g_2(x_0 - z_0, r + n) \right] \mathrm{d}n \right\}$$

$$+ \underline{\omega} \left\{ \int_0^r \left[f_2(z_0, n) g_1(x_0 - z_0, r - n) - \sum_{k=0}^{m} (-1)^k \left(\frac{2k}{m} - 1\right) f_1^{(k)}(z_0, n) g_2(x_0 - z_0, r - n) \right] \mathrm{d}n \right.$$

$$+ \int_r^{+\infty} \left[f_2(z_0, n) g_1(x_0 - z_0, n - r) + \sum_{k=0}^{m} (-1)^k \left(\frac{2k}{m} - 1\right) f_1^{(k)}(z_0, n) g_2(x_0 - z_0, n - r) \right] \mathrm{d}n$$

$$\left. - \int_0^{+\infty} \left[f_2(z_0, n) g_1(x_0 - z_0, r + n) + \sum_{k=0}^{m} (-1)^k \left(\frac{2k}{m} - 1\right) f_1^{(k)}(z_0, n) g_2(x_0 - z_0, r + n) \right] \mathrm{d}n \right\}$$

$$\mathrm{d}z_0 .$$

UNIVERSITEIT GENT

Mustard convolutions

Writing

$$f^{even}(x_0, r) = \begin{cases} f(x_0, r) & r > 0 \\ f(x_0, -r) & r < 0 \end{cases},$$

$$f^{odd}(x_0, r) = \begin{cases} f(x_0, r) & r > 0 \\ -f(x_0, -r) & r < 0 \end{cases}$$

we finally get

$$\mathcal{F}^{-1}\left(\mathcal{F}(f)\mathcal{F}(g)\right)(\mathbf{x}) = \frac{-i}{4\pi c} \times$$

$$\Bigg(\quad \left\{ f_1^{even} \star g_1^{even}(x_0, r) + \sum_{k=0}^{m} (-1)^k \left(\frac{2k}{m} - 1\right) f_2^{odd,(k)} \star g_2^{odd}(x_0, r) \right\}$$

$$+ \underline{\omega} \left\{ f_2^{odd} \star g_1^{even}(x_0, r) - \sum_{k=0}^{m} (-1)^k \left(\frac{2k}{m} - 1\right) f_1^{even,(k)} \star g_2^{odd}(x_0, r) \right\} \Bigg)$$

The expression for the right Mustard convolution, defined as $f \star_M^R g = \mathcal{F}_S^{-1}\left(\mathcal{F}_S(g)\mathcal{F}_S(f)\right)$, is obtained by interchanging $f \leftrightarrow g$.

Generalised translation

In classical Fourier theory, the Fourier transform of a function translated over t is given by:

$$\mathcal{F}(\tau_t f)(y) = \int_{-\infty}^{+\infty} \frac{e^{-ixy}}{\sqrt{2\pi}} \tau_t f(x) \mathrm{d}x = e^{-ity} \int_{-\infty}^{+\infty} \frac{e^{-ixy}}{\sqrt{2\pi}} f(x) \mathrm{d}x$$

$$= \sqrt{2\pi}\, K(t,y) \int_{-\infty}^{+\infty} (K(x,y))^* f(x) \mathrm{d}x$$

and therefore

$$\tau_t f(x) = \sqrt{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} K(y,x) \left(K(t,y)\right)^* \left(K(u,y)\right)^* f(u)\, \mathrm{d}y \mathrm{d}u.$$

Generalised translation

In classical Fourier theory, the Fourier transform of a function translated over t is given by:

$$\mathcal{F}(\tau_t f)(y) = \int_{-\infty}^{+\infty} \frac{e^{-ixy}}{\sqrt{2\pi}}\, \tau_t f(x) \mathrm{d}x = e^{-ity} \int_{-\infty}^{+\infty} \frac{e^{-ixy}}{\sqrt{2\pi}}\, f(x) \mathrm{d}x$$

$$= \sqrt{2\pi}\, K(t,y) \int_{-\infty}^{+\infty} (K(x,y))^* f(x) \mathrm{d}x$$

and therefore

$$\tau_t f(x) = \sqrt{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} K(y,x)\, (K(t,y))^*\, (K(u,y))^*\, f(u)\, \mathrm{d}y \mathrm{d}u.$$

We therefore define the generalised translation $\tau_{\mathbf{y}}$ for the slice Fourier transform as

$$\tau_{\mathbf{y}} f(\mathbf{x}) = \int K(\mathbf{z},\mathbf{x})\, (K(\mathbf{y},\mathbf{z}))^*\, (K(\mathbf{u},\mathbf{z}))^*\, f(\mathbf{u})\, \mathrm{d}\mathbf{z}\, \mathrm{d}\mathbf{u}.$$

Generalised translation

After lengthy calculations, the expression for the generalised translation reads

$$
\begin{aligned}
T_{\mathbf{y}} f(\mathbf{x}) = & \frac{-i\Gamma\left(\frac{m}{2}\right)}{8c\pi^{m/2+1}} \times \\
& \Bigg\{ \left[f_1^{even}(x_0 - y_0, r - g) - \sum_{k=0}^{m} (-1)^k \left(\frac{2k}{m} - 1 \right) f_2^{odd,(k)}(x_0 - y_0, r - g)\underline{\eta} \right] \\
& + \left[f_1^{even}(x_0 - y_0, r + g) + \sum_{k=0}^{m} (-1)^k \left(\frac{2k}{m} - 1 \right) f_2^{odd,(k)}(x_0 - y_0, r + g)\underline{\eta} \right] \\
& + \underline{\omega} \left[f_2^{odd}(x_0 - y_0, r - g) + \sum_{k=0}^{m} (-1)^k \left(\frac{2k}{m} - 1 \right) f_1^{even,(k)}(x_0 - y_0, r - g)\underline{\eta} \right] \\
& + \underline{\omega} \left[f_2^{odd}(x_0 - y_0, r + g) - \sum_{k=0}^{m} (-1)^k \left(\frac{2k}{m} - 1 \right) f_1^{even,(k)}(x_0 - y_0, r + g)\underline{\eta} \right] \Bigg\}
\end{aligned}
$$

UNIVERSITEIT GENT

Generalised translation

After lengthy calculations, the expression for the generalised translation reads

$$T_{\mathbf{y}} f(\mathbf{x}) = \frac{-i\Gamma\left(\frac{m}{2}\right)}{8c\pi^{m/2+1}} \times$$

$$\Bigg\{ \left[f_1^{even}(x_0 - y_0, r - g) - \sum_{k=0}^{m} (-1)^k \left(\frac{2k}{m} - 1 \right) f_2^{odd,(k)}(x_0 - y_0, r - g) \underline{\eta} \right]$$

$$+ \left[f_1^{even}(x_0 - y_0, r + g) + \sum_{k=0}^{m} (-1)^k \left(\frac{2k}{m} - 1 \right) f_2^{odd,(k)}(x_0 - y_0, r + g) \underline{\eta} \right]$$

$$+ \underline{\omega} \left[f_2^{odd}(x_0 - y_0, r - g) + \sum_{k=0}^{m} (-1)^k \left(\frac{2k}{m} - 1 \right) f_1^{even,(k)}(x_0 - y_0, r - g) \underline{\eta} \right]$$

$$+ \underline{\omega} \left[f_2^{odd}(x_0 - y_0, r + g) - \sum_{k=0}^{m} (-1)^k \left(\frac{2k}{m} - 1 \right) f_1^{even,(k)}(x_0 - y_0, r + g) \underline{\eta} \right] \Bigg\}$$

By definition the corresponding convolution equals the expression for the left Mustard convolution:

$$\int_{\mathbf{y}} T_{\mathbf{y}} f(\mathbf{x}) g(\mathbf{y}) \mathrm{d}\mathbf{y} = \mathcal{F}_S^{-1} \left(\mathcal{F}_S(f) \mathcal{F}_S(g) \right)(\mathbf{x})$$

Generalised translation

After lengthy calculations, the expression for the generalised translation reads

$$
\begin{aligned}
T_{\mathbf{y}} f(\mathbf{x}) = & \frac{-i\Gamma\left(\frac{m}{2}\right)}{8c\pi^{m/2+1}} \times \\
& \Bigg\{ \left[f_1^{even}(x_0 - y_0, r - g) - \sum_{k=0}^{m} (-1)^k \left(\frac{2k}{m} - 1\right) f_2^{odd,(k)}(x_0 - y_0, r - g)\underline{\eta} \right] \\
& + \left[f_1^{even}(x_0 - y_0, r + g) + \sum_{k=0}^{m} (-1)^k \left(\frac{2k}{m} - 1\right) f_2^{odd,(k)}(x_0 - y_0, r + g)\underline{\eta} \right] \\
& + \underline{\omega} \left[f_2^{odd}(x_0 - y_0, r - g) + \sum_{k=0}^{m} (-1)^k \left(\frac{2k}{m} - 1\right) f_1^{even,(k)}(x_0 - y_0, r - g)\underline{\eta} \right] \\
& + \underline{\omega} \left[f_2^{odd}(x_0 - y_0, r + g) - \sum_{k=0}^{m} (-1)^k \left(\frac{2k}{m} - 1\right) f_1^{even,(k)}(x_0 - y_0, r + g)\underline{\eta} \right] \Bigg\}
\end{aligned}
$$

By definition the corresponding convolution equals the expression for the left Mustard convolution:

$$
\int_{\mathbf{y}} T_{\mathbf{y}} f(\mathbf{x}) g(\mathbf{y}) \mathrm{d}\mathbf{y} = \mathcal{F}_S^{-1}\left(\mathcal{F}_S(f)\mathcal{F}_S(g)\right)(\mathbf{x})
$$

$$
\text{though still } \neq \mathcal{F}_S^{-1}\left(\mathcal{F}_S(g)\mathcal{F}_S(f)\right)(\mathbf{x})
$$

Conclusions and further research

Conclusions

- Clifford-Hermite functions
- Closed form for kernel function
- Clifford convolutions

Future research

- Cauchy-Kovalevskaya extension
- Segal-Bargmann transform