# Orthogonal instantons and skew-Hamiltonian matrices

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# The moduli space $M(r, n)$

$M(r,n)$= the moduli space of (slope) stable v.b. on $\mathbb{P}^2$ with Chern classes $(0,n)$ and rank $2 \le r \le n$. ($M(r,n) = \emptyset$ if $r > n$.)

$E \in M(r,n)$ is the cohomology bundle of the monad:

$$I \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{g} V^* \otimes \Omega^1_{\mathbb{P}^2}(2) \xrightarrow{f} V \otimes \mathcal{O}_{\mathbb{P}^2}(1)$$

$V = \mathrm{H}^1(E(-1))$ has dim $n$ (independently from $r$!), $I = \mathrm{H}^1(E(-3))$ has dim $n - r$, $f \in U \otimes V \otimes V$ is the natural multiplication map, $\mathbb{P}^2 = \mathbb{P}(U)$.

Rem: equivalently $E$ is the cohomology of a *linear monad*:

$$V^* \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\alpha} K \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{\beta} V \otimes \mathcal{O}_{\mathbb{P}^2}(1)$$

where $K = \mathrm{H}^1(E \otimes \Omega^1_{\mathbb{P}^2})$ has dim $2n + r$.

So all elements of $M(r,n)$ are "generalized instantons".

# Orthogonal and symplectic bundles

- An orthogonal v.b. is a pair $(E, \alpha)$ consisting of a v.b. $E$ and an iso $\alpha : E \to E^*$ with ${}^t\alpha = \alpha$.
- A symplectic v.b. is a pair $(E, \alpha)$ as above, with ${}^t\alpha = -\alpha$.
- A bundle with no additional structure is called unstructured.

**Lemma.** The map $f$ encodes all the info: $E(f)$ and $E(f')$ simple are iso iff $f$ and $f'$ are $\mathrm{SL}(V)$-equivalent.

We can recover the structure of the v.b. from $f \in U \otimes V \otimes V$.

**Proposition.** The bundle $E(f)$ is:

- orthogonal iff the map $f \in U \otimes \Lambda^2 V$;
- symplectic iff the map $f \in U \otimes S^2 V$.

# Unstructured and symplectic bundles

$M(r,n)$= the moduli space of stable unstructured v.b. on $\mathbb{P}^2$ with Chern classes $(0,n)$ and rank $2 \le r \le n$.

**Theorem** [Hulek, 1980] When non-empty, $M(r,n)$ is a smooth irreducible variety of dimension $2rn - r^2 + 1$.

$M_{sp}(r,n)$ = the moduli space of stable symplectic v.b. on $\mathbb{P}^2$ with Chern classes $(0,n)$ and rank $2 \le r \le n$.

**Theorem** [Ottaviani, 2007] When non-empty, $M_{sp}(r,n)$ is a smooth irreducible variety of dimension $(r+2)n - \binom{r+1}{2}$.

Rem: in particular $r$ is even in the symplectic case.

# Orthogonal bundles

$M_{ort}(r, n)$ = the moduli space of stable orthogonal v.b. on $\mathbb{P}^2$ with Chern classes $(0, n)$ and rank $3 \leq r \leq n$.

**Wish.** When non-empty, $M_{ort}(r, n)$ is a smooth **irreducible** variety of dimension $(r-2)n - \binom{r}{2}$.

In fact the smoothness argument generalizes "smoothly":

**Proposition.** When non-empty, the moduli space $M_{ort}(r, n)$ is smooth of dimension $(r-2)n - \binom{r}{2}$.

On the contrary, the irreducibility argument **does not**.
Why? What goes wrong? What goes right?
(Results from a joint project with R. Abuaf.)

# Bundles with trivial splitting on a line

Notation: $M_\star(r,n)$ for $\star = \emptyset$, *sp*, *ort*.

$$M^0_\star(r,n) = \{E \in M_\star(r,n) \mid E_{|_\ell} = \mathcal{O}^r_{\mathbb{P}^1} \text{ for some line } \ell\}$$

By semicontinuity, if $E_{|_\ell}$ is trivial on a line $\ell$, then it is trivial on the general line.

Proving irreducibility of $M^0_\star(r,n)$ is easier!
Then, using a deformation argument due to Hirschowitz, we get:

- $M(r,n) = \overline{M^0(r,n)}$ is irreducible [Hulek, 1980]
- $M_{sp}(r,n) = \overline{M^0_{sp}(r,n)}$ is irreducible [Ottaviani, 2007]
- $M_{ort}(r,n) \neq \overline{M^0_{ort}(r,n)}$

# Degeneration arguments and the Mumford invariant

When we restrict an unstructured or a symplectic bundle to $\mathbb{P}^1$, the only rigid bundle is the trivial one, $\mathcal{O}_{\mathbb{P}^1}^r$ [Ramanathan, 1983].

In the orthogonal case there are 2 rigid bundles:

$$\mathcal{O}_{\mathbb{P}^1}^r \quad \text{and} \quad \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}^{r-2} \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

- If $E$ is an orthogonal v.b. on $\mathbb{P}^1$, then $\mathrm{h}^1(E(-1)) \bmod 2$ is invariant under deformations. [Mumford, 1971]
- Orthogonal rk 2 v.b. on $\mathbb{P}^1$ are rigid; for rk $\geq 3$ the *Mumford invariant* is the only one. Two such v.b. can be deformed into each other iff they have the same Mumford inv. [Hulek, 1981]

Remember that for us $\mathrm{h}^1(E(-1)) = n = c_2(E)$.

**Proposition.** If $E \in M^0_{ort}(r, n)$, then $n$ is even.

# A closer look

If $P, Q$ and $R$ the 3 "slices" of $f$, then we can write:

$$\mathrm{H}^0(f) = \begin{bmatrix} 0 & P & Q \\ -P & 0 & R \\ -Q & -R & 0 \end{bmatrix}$$

$P, Q$ and $R$ are $n \times n$ unstructured, symmetric, and skew-symmetric matrices for $\star = \emptyset$, $sp$, and $ort$ respectively.

**Lemma.** $E \in M^0_\star(r, n)$ iff $Q$ is invertible.

Rem: that's why $n$ is even in the orthogonal case!

We get a link between $r = \mathrm{rk}\, E$ and the 3 matrices $P$, $Q$, and $R$:

$$\mathrm{rk}(PQ^{-1}R - RQ^{-1}P) = r$$

# Irreducibility results

Using a standard fibration argument (so standard that we skip it):

**Theorem.** Let $n$ and $3 \leq r \leq n$ be two positive integers, $n$ even. Let V be a complex v.s. of dimension $n$, and let $J$ denote the standard symplectic form. If the variety:

$$\mathcal{C}_{r,n} = \{(A, B) \in \Lambda^2 V \times \Lambda^2 V \mid \mathrm{rk}(AJB - BJA) \leq r\}$$

is irreducible, then the same is true for the moduli space $M^0_{ort}(r, n)$.

**Key Lemma.** $\mathcal{C}_{r,n}$ is irreducible for $r = n$ and $n \geq 4$, and for $r = n - 1$ and $n \geq 8$.

# Skew-Hamiltonian matrices

What happens for $M^0(r,n)$ and $M^0_{sp}(r,n)$? With similar arguments one reduces to proving the irreducibility of:

$$\{(A,B) \in (V \otimes V) \times (V \otimes V) \mid \mathrm{rk}[A,B] \leq r\}$$

and of: $\{(A,B) \in S^2 V \times S^2 V \mid \mathrm{rk}[A,B] \leq r\}$ resp.

So we want commutators too!

Notice that $\mathrm{rk}(AJB - BJA) = \mathrm{rk}[JA, JB]$.
Enter the scene: skew-Hamiltonian matrices.

A *skew-Hamiltonian* matrix is of the form $JB$, $B \in \Lambda^2 V$.

Rem: skew-Hamiltonians are the less cool cousins of Hamiltonians ($JS$, $S \in S^2 V$), which correspond to the Lie algebra of $\mathrm{Sp}(n)$.

# Regular elements

Fix $B \in \Lambda^2 V$ and consider:

$$\varphi^B : \Lambda^2 V \to S^2 V \qquad A \mapsto AJB - BJA$$

If the intersection $\operatorname{Im} \varphi^B \cap S^2 V_{\leq r}$ of $\operatorname{Im} \varphi^B$ with the determinantal variety $S^2 V_{\leq r}$ is irreducible for a “good” $B$, the Key Lemma follows! (Modulo more fibration arguments.)

For $M^0(r, n)$ and $M^0_{sp}(r, n)$ “good” means “regular in the Lie algebra”, an element whose commutator has minimal dimension.

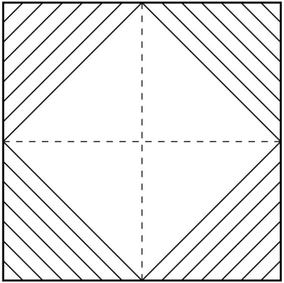
So we want regular matrices too!
Alas, the notion of regular element is meaningless for us. But we can use the actions of $\mathrm{Sp}(n)$ on $\Lambda^2 V$ and on skew-Hamiltonians.

# Diamond matrices

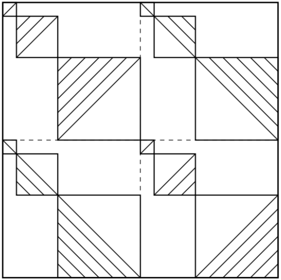
Using $\mathrm{Sp}(n)$ we define our "regular elements", estimate the dimension and also get **explicit equations** of $\operatorname{Im}\varphi^B$!

**Proposition.**[Noferini, 2013] Let $B \in \Lambda^2 V$ s.t. $JB$ is a regular skew-Hamiltonian, and let $A \in \Lambda^2 V$ any skew-symmetric matrix. Then $AJB - BJA$ is symplectically congruent to a *diamond matrix*.

What is a diamond matrix?



A diamond matrix corresponding to the partition $\underline{d} = (\frac{n}{2})$.



A diamond matrix corresponding to a partition $\underline{d} = (d_1, d_2, d_3)$.

# Irreducibility results and conjectural bounds

With some elbow grease we finally prove irreducibility, by means of a strong connectedness result.

**Lemma.** If $JB$ is regular, then the intersection $\operatorname{Im}\varphi^B \cap S^2V_{\leq r}$ is irreducible of dimension $nr - \frac{3}{2}n - \binom{r}{2}$ for $r = n$ and $n \geq 4$ and for $r = n-1$ and $n \geq 8$.

In fact we believe that more is true:

**Conjecture.** Irreducibility of $M^0_{ort}(r,n)$ holds for any $6r - 5n \geq 2$.

Rem: For $r = n$ and $n-1$ we re-obtain what we just proved.

# Open questions: the case $c_2$ odd

In the case $c_2$ odd it appears that (almost) none of our techniques apply. Some remarks from a WIP with M. Jardim and S. Marchesi:

- Orthogonal bundles with odd $c_2$ cannot have trivial splitting on the general line, and they do not deform to ones that do.
- In fact for $n$ odd $M_{ort}(r,n) = \overline{M^1_{ort}(r,n)}$, where:

  $$M^1_{ort}(r,n) : \{E \mid E_{|\ell} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}^{r-2} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \text{ for some line } \ell\}$$

  We call these bundles *weakly framed*.
- The moduli space is not empty! $(S^2\,\mathrm{T}\,\mathbb{P}^2)(-3)$, is an example of a weakly framed stable rk 3 orthogonal bundle on $\mathbb{P}^2$ with Chern classes $(0,3)$.

# A curious result and one last question

**Proposition.** Let $V$ be a complex vector space of even dimension $n$. Define $\phi$ as:

$$\phi : \mathbb{P}(\Lambda^2 V \times \Lambda^2 V) \dashrightarrow \mathbb{P}(S^2 V) \qquad ([A],[B]) \mapsto [AJB - BJA].$$

a) For $n = 2$, $\phi$ is not defined.

b) For $n = 4$, $\operatorname{Im}\phi$ is a $\mathbb{G}(2,5)$ in $\mathbb{P}^9 = \mathbb{P}(S^2\mathbb{C}^4)$.

c) For $n = 6$, $\operatorname{Im}\phi$ is a hypersurface of deg 4 in $\mathbb{P}^{20} = \mathbb{P}(S^2\mathbb{C}^6)$.

d) For $n \geq 8$, $\phi$ is locally of maximal rank. (So $\phi$ is dominant.)

**Question.** Is there a value of $r$ for which $\phi$ composed with the projection $\mathbb{P}(S^2 V) \twoheadrightarrow \mathbb{P}(S^2 V_{\leq r})$ is surjective? This is false for $r = 2$ [Noferini, 2013], but it remains open for higher values of $r$.

Thank you :)