Weak multiplier bialgebra

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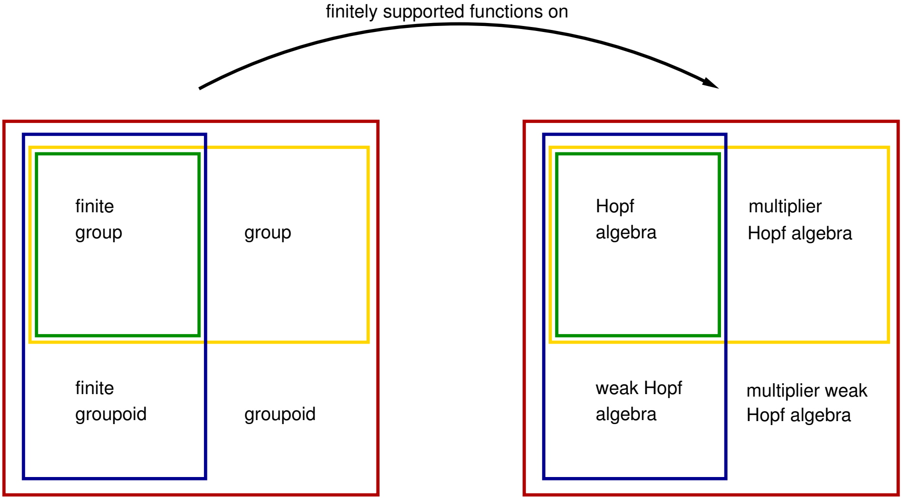
Based on [GB, J Gómez-Torrecillas & E López-Centella, TAMS 2015].

Motivation


finitely supported functions on
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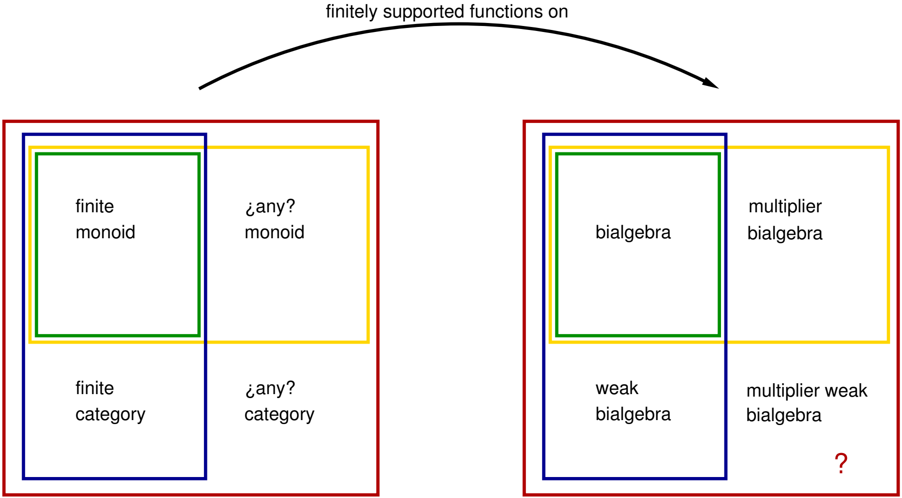
Motivation


finitely supported functions on
finite monoid
¿any? monoid
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bialgebra
multiplier bialgebra
weak bialgebra
multiplier weak bialgebra
?

The group example

Let G be a group with multiplication $m : G \times G \to G$ and unit $e \in G$

k a field

$k(G)$ the unital algebra of all k-valued functions on G

$k[G]$ the non-unital algebra of finitely supported k-valued functions on G.

If G is finite then $k \overset{e^*}{\leftarrow} k[G] \overset{m^*}{\to} k[G \times G] \cong k[G] \otimes k[G]$ is a Hopf algebra.

In general, $k(G) \overset{m^*}{\to} k(G \times G)$ is determined

- by its restriction $\Delta := m^*|_{k[G]} \colon k[G] \to k(G \times G)$
- equivalently – due to density of $k[G] \otimes k[G] \cong k[G \times G]$ in $k(G \times G)$ – by unique maps $T_{1,2} : k[G] \otimes k[G] \to k[G] \otimes k[G]$ s.t.

$$k[G] \otimes k[G] \overset{T_1}{\to} k[G] \otimes k[G] \rightarrowtail k(G \times G) \ : f' \otimes f \mapsto \Delta(f')(1 \otimes f)$$

$$k[G] \otimes k[G] \overset{T_2}{\to} k[G] \otimes k[G] \rightarrowtail k(G \times G) \ : f' \otimes f \mapsto (f' \otimes 1)\Delta(f),$$

and $k(G) \overset{e^*}{\to} k$ is determined by $\varepsilon := e^*|_{k[G]} \colon k[G] \to k$.

This does not make $k[G]$ a Hopf algebra.

Abstraction of a multiplier valued comultiplication

[Fons Van Daele]

$k[G]$	$\rightsquigarrow$	a k-algebra A with non-unital but surjective & non-degenerate multiplication
$k(G)$	$\rightsquigarrow$	the multiplier algebra $\mathbb{M}(A)$ of A: $\mathbb{M}(A) := \{\lambda, \varrho : A \to A \mid a\lambda(b) = \varrho(a)b,\ \forall a, b \in A\}$ it is a unital algebra via $(\lambda', \varrho')(\lambda, \varrho) = (\lambda' \circ \lambda, \varrho \circ \varrho')$ $A \to \mathbb{M}(A)$, $a \mapsto (a(-), (-)a)$ makes A a dense ideal
Δ	$\rightsquigarrow$	$\Delta : A \to \mathbb{M}(A \otimes A)$, equivalently,
$T_{1,2}$	$\rightsquigarrow$	$T_{1,2} : A \otimes A \to A \otimes A$
ε	$\rightsquigarrow$	$\varepsilon : A \to k$

suitable axioms on them define a multiplier Hopf algebra.

multiplier Hopf algebra with unit $=$ Hopf algebra

The groupoid example

Let G/S be a groupoid with multiplication $m : G \times_S G \to G$
k a field
$k(G)$ the unital algebra of all k-valued functions on G
$k[G]$ the non-unital algebra of finitely supported k-valued functions on G.

If G is finite then $k[G] = k(G)$ is a weak Hopf algebra.

In general, the characteristic function E of $G \times_S G$ is an idempotent in $k(G \times G)$
$\Rightarrow$ $k(G \times G) \twoheadrightarrow k(G \times_S G)$ is split by $E(-)$

$\Rightarrow$ there is a map $k(G) \xrightarrow{m^*} k(G \times_S G) \rightarrowtail^{E(-)} k(G \times G)$, determined

- by its restriction $\Delta : k[G] \to k(G \times G)$, equivalently,
- by unique maps $T_{1,2} : k[G] \otimes k[G] \to k[G] \otimes k[G]$ s.t.

$$k[G] \otimes k[G] \xrightarrow{T_1} k[G] \otimes k[G] \rightarrowtail k(G \times G) \ : f' \otimes f \mapsto \Delta(f')(1 \otimes f)$$

$$k[G] \otimes k[G] \xrightarrow{T_2} k[G] \otimes k[G] \rightarrowtail k(G \times G) \ : f' \otimes f \mapsto (f' \otimes 1)\Delta(f),$$

and there is $\varepsilon : k[G] \to k$, $f \mapsto \sum_{x \in S} f(x)$.

This does not make $k[G]$ a weak Hopf algebra.

The occurrence of the idempotent multiplier E

[Fons Van Daele & Shuanhong Wang]

$k[G]$	$\rightsquigarrow$	a k-algebra A with non-unital but surjective & non-degenerate multiplication
$k(G)$	$\rightsquigarrow$	the multiplier algebra $\mathbb{M}(A)$
E	$\rightsquigarrow$	$E \in \mathbb{M}(A \otimes A)$ s.t. $E^2 = E$
Δ	$\rightsquigarrow$	$\Delta : A \to \mathbb{M}(A \otimes A)$, equivalently,
$T_{1,2}$	$\rightsquigarrow$	$T_{1,2} : A \otimes A \to A \otimes A$
ε	$\rightsquigarrow$	$\varepsilon : A \to k$

suitable axioms on them define a weak multiplier Hopf algebra.

weak multiplier Hopf algebra with unit $\supsetneq_{?}$ weak Hopf algebra

Weak multiplier bialgebra

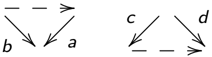
[GB, Pepe Gómez-Torrecillas & Esperanza López-Centella]

- a non-unital k-algebra A with surjective & non-degenerate multiplication
- $E \in \mathbb{M}(A \otimes A)$ s.t. $E^2 = E$
- $\Delta : A \to \mathbb{M}(A \otimes A)$, equivalently,
  $T_{1,2} : A \otimes A \to A \otimes A$
- $\varepsilon : A \to k$

suitable axioms on them define a weak multiplier bialgebra.

weak multiplier bialgebra with unit $=$ weak bialgebra
weak multiplier Hopf algebra $\not\subseteq$ weak multiplier bialgebra
?

Examples

- weak bialgebras
- partial bialgebras [De Commer & Timmermann]
- direct sums of weak multiplier bialgebras
- linear spans of arbitrary categories
- finitely supported functions on "slice finite" categories (i.e. on categories in which there are finitely many triangles 

with given arrows a, b, c, d)
- regular weak multiplier Hopf algebras in [Van Daele & Wang]

The base algebras

For any weak multiplier bialgebra A, there are maps

$$\overline{\sqcap}^R : A \to \mathbb{M}(A), \qquad a \mapsto ((\mathrm{id} \otimes \varepsilon)(T_1(- \otimes a)), (\mathrm{id} \otimes \varepsilon)(E(- \otimes a)))$$
$$\overline{\sqcap}^L : A \to \mathbb{M}(A), \qquad a \mapsto ((\varepsilon \otimes \mathrm{id})((a \otimes -)E), (\varepsilon \otimes \mathrm{id})(T_2(a \otimes -))) \ .$$

Theorem. Their images $R := \overline{\sqcap}^R(A)$ and $L := \overline{\sqcap}^L(A)$ are commuting non-unital subalgebras of $\mathbb{M}(A)$.

Multiplier bialgebra

Theorem. For a weak multiplier bialgebra A, tfae.

- $E = 1$ in $\mathbb{M}(A \otimes A)$.
- $\varepsilon(ab) = \varepsilon(a)\varepsilon(b) \quad \forall a, b$.
- $\overline{\sqcap}^L(a) = \varepsilon(a)1$ in $\mathbb{M}(A) \quad \forall a$.
- $\overline{\sqcap}^R(a) = \varepsilon(a)1$ in $\mathbb{M}(A) \quad \forall a$.

A weak bialgebra with these equivalent properties is precisely a bialgebra. Hence such a weak multiplier bialgebra could be termed a multiplier bialgebra.

Regularity and fullness

Definition. A weak multiplier bialgebra A is regular if also A^{op} is a weak multiplier bialgebra (with the same Δ, E, ε).

In a regular weak multiplier bialgebra A, there are further two maps

$$\sqcap^L : A \to \mathbb{M}(A), \quad a \mapsto ((\varepsilon \otimes \mathrm{id})(E(a \otimes -)), (\varepsilon \otimes \mathrm{id})(\Delta(-)(a \otimes 1)))$$
$$\sqcap^R : A \to \mathbb{M}(A), \quad a \mapsto ((\mathrm{id} \otimes \varepsilon)((1 \otimes a)\Delta(-)), (\mathrm{id} \otimes \varepsilon)((- \otimes a)E)) \ .$$

Theorem. For a regular weak multiplier bialgebra A, tfae.

- $\overline{\sqcap}^R(A) = \sqcap^R(A)$.
- $\langle (\mathrm{id} \otimes \varepsilon)(\Delta(a)(1 \otimes b)) | a, b \in A \rangle = A$.
- $\langle (\mathrm{id} \otimes \varepsilon)((1 \otimes b)\Delta(a)) | a, b \in A \rangle = A$.
- $\langle (\mathrm{id} \otimes \omega)(\Delta(a)(1 \otimes b)) | a, b \in A, \omega : A \to k \rangle = A$.
- $\langle (\mathrm{id} \otimes \omega)((1 \otimes b)\Delta(a)) | a, b \in A, \omega : A \to k \rangle = A$.

Then we say that Δ is right full.

There is a symmetric notion of left full Δ.

The structure of the base algebras

Theorem. If the comultiplication of a regular weak multiplier bialgebra A is right full, then for $R := \overline{\sqcap}^R(A) = \sqcap^R(A)$ the following hold.

- R is a non-unital subalgebra of $\mathbb{M}(A)$.
- R possesses a coalgebra structure.
- The comultiplication is a bimodule section of the multiplication. Equivalently, the multiplication is a bicomodule section of the comultiplication (i.e. the coalgebra R is coseparable).
- The algebra R is firm (i.e. the quotient of the multiplication to $R \otimes_R R \to R$ is invertible); in fact it has local units.

$L := \overline{\sqcap}^L(A) = \sqcap^L(A)$ has the same structures if the comultiplication is left full.

This generalizes the separable-Frobenius structure of the base algebras of a weak bialgebra.

Theorem. If the comultiplication of a regular weak multiplier bialgebra A is left and right full, then R and L are anti-isomorphic as algebras and coalgebras.

Modules and comodules

Definition. [Quillen] A module M over a non-unital algebra R is firm if the quotient $M \otimes_R R \to M$ of the action is invertible.

If R is a firm algebra, then the R-bimodules which are firm on both sides constitute a monoidal category FirmBim(R) via $\otimes_R$.

Theorem. If the comultiplication of a regular weak multiplier bialgebra A is right full, then those right A-modules whose action is surjective and non-degenerate constitute a monoidal category admitting a strict monoidal faithful functor to FirmBim(R) (where $R := \overline{\sqcap}^R(A) = \sqcap^R(A)$).

Theorem. [GB Int J Math 2014] If the comultiplication of a regular weak multiplier bialgebra A is right full, then there is a suitable notion of full right A-comodule; such that their category is monoidal admitting a strict monoidal faithful functor to FirmBim(R) (where $R := \overline{\sqcap}^R(A) = \sqcap^R(A)$).

Antipode

Theorem. For a regular weak multiplier bialgebra A, there is a bijection between

- weak (aka Von Neumann) inverses $R_{1,2}$ of $T_{1,2} : A \otimes A \to A \otimes A$ (wrt initial and final idempotents fixed in terms of E)
- a map $S : A \to \mathbb{M}(A)$ – called the 'antipode' – obeying three axioms expressing that S is a generalized weak 'convolution inverse' of $A \rightarrowtail \mathbb{M}(A)$.

The antipode axioms imply

$$m \circ (S \otimes \mathrm{id}) \circ T_1 = m \circ (\sqcap^R \otimes \mathrm{id}) \quad \text{and} \quad m \circ (\mathrm{id} \otimes S) \circ T_2 = m \circ (\mathrm{id} \otimes \sqcap^L)$$

cf. weak Hopf algebra axioms.

Notions of weak multiplier Hopf algebra

regular wmba

$\cup$

regular wmha in [Van Daele & Wang] $\subset$ regular wmba with antipode $\subsetneq_?$ wmha in [Van Daele & Wang]

$\cup$

wha in [B, Nill & Szlachányi]

Thank you!