Symmetries of string compactifications and generalization of Freed-Witten anomaly

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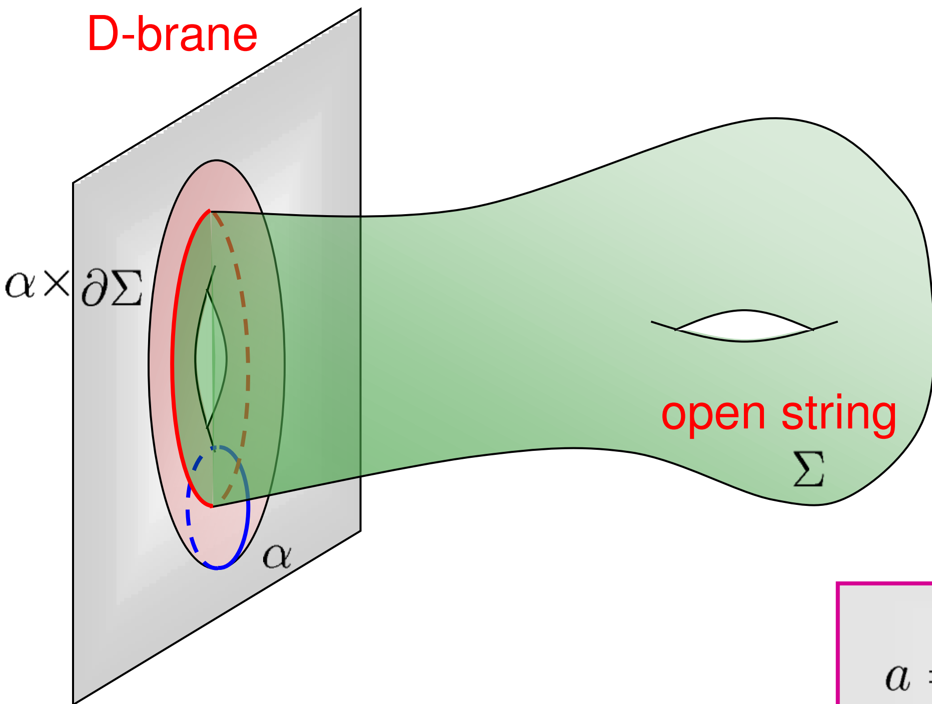
S.A., S.Banerjee arXiv:1405.0291

Freed-Witten anomaly

Freed,Witten '99



worldsheet Dirac operator

U(1) gauge field on the brane

$$\mathrm{pfaff}(D)\ \exp\left(2\pi \mathrm{i} \oint_{\partial\Sigma} A\right)$$

going around a loop in the moduli space

$$(-1)^a \mathrm{pfaff}(D)$$

$$a = \int_{\alpha\times\partial\Sigma} w_2(\text{D-brane})$$

Stiefel–Whitney class

path integral is well-defined

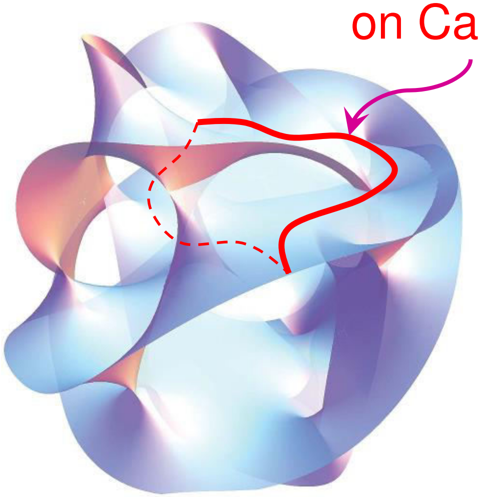
under variation of moduli periods of gauge field can get *half-integer* shifts

$$\oint_{\partial\Sigma} A \mapsto \oint_{\partial\Sigma} A + \frac{a}{2}$$

A is not an ordinary U(1) gauge field, but a Spin^c connection

Our result: a new anomaly

a non-trivial cycle on Calabi-Yau



gauge potentials of type IIB string theory

$$C = \int_\gamma A^{\text{even}} e^{-B}$$

RR-moduli of the HM moduli space

monodromy transformation around large volume

large gauge transformation of the *B*-field

$$\int_{\gamma^a} B \mapsto \int_{\gamma^a} B + \epsilon^a$$

symplectic transformation + anomaly

$$C \;\mapsto\; \rho(M_{\epsilon^a}) \cdot C \;+\; \text{half-integer terms}$$

axionic couplings of

D3-branes (wrapping 4-cycles)

D5-branes (wrapping CY)

$$C_a \;\mapsto\; \text{hom. terms} \;+\; A_{ab}\epsilon^b$$

$$C_0 \;\mapsto\; \text{hom. terms} \;+\; \frac{c_{2,a}}{8}\,\epsilon^a - \frac{1}{2}\,A_{ab}\epsilon^a\epsilon^b$$

second Chern class of CY

half-integer matrix

$$A_{ab} = \frac{1}{2}\kappa_{abb} + \mathbb{Z}$$

Calabi-Yau compactifications

Type II string theory

compactification on a Calabi-Yau ↓

$\mathcal{N}$=2 supergravity in 4d coupled to matter

- supergravity multiplet (metric & graviphoton)
- vector multiplets (gauge fields & scalars)
- hypermultiplets (only scalars)

The low energy effective action is completely determined by the geometry of **moduli space**

$$\mathcal{M}_{\rm VM} \times \mathcal{M}_{\rm HM}$$

In type IIB

$\mathcal{M}_{\rm VM}$ – special Kähler manifold parametrized by complex structure moduli

$\mathcal{M}_{\rm HM}$ – quaternion-Kähler manifold parametrized by

$v^a = b^a + \mathrm{i}t^a$	complexified Kähler moduli
$c^0, c^a, \tilde{c}_a, \tilde{c}_0$	periods of RR gauge potentials
ψ	NS-axion (dual to the B-field)
τ_2	dilaton (string coupling $\tau_2 \sim g_s^{-1}$)

$a = 1, \dots, h^{1,1}$

$\dim \mathcal{M}_{\rm HM} = 4(h^{1,1}+1)$

$c^0 \equiv \tau_1$

↓

$\tau = \tau_1 + \mathrm{i}\tau_2$

axio-dilaton

Classical symmetries

In the classical limit ($\alpha' \to 0,\ g_s \to 0$) the metric on $\mathcal{M}_{\rm HM}$ is given by the *c-map.*
It has the following *continuous isometries*:

determined by the prepotential $F(X)$ describing Kähler moduli

- **S-duality group SL(2,ℝ)**

$$\tau \mapsto \frac{a\tau+b}{c\tau+d} \qquad t^a \mapsto t^a|c\tau+d| \qquad \tilde c_a \mapsto \tilde c_a$$

$$\begin{pmatrix} c^a \\ b^a \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} c^a \\ b^a \end{pmatrix} \qquad \begin{pmatrix} \tilde c_0 \\ \psi \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}\begin{pmatrix} \tilde c_0 \\ \psi \end{pmatrix}$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- **Heisenberg symmetry**

generated by shifts of NS-axion $k = \partial_\psi$
and RR-fields

$$h_{\eta^\Lambda,\tilde\eta_\Lambda} = \eta^\Lambda \partial_{c^\Lambda} + \tilde\eta^\Lambda \partial_{\tilde c^\Lambda} + \frac{1}{2}(\tilde\eta_\Lambda c^\Lambda - \eta^\Lambda \tilde c_\Lambda)\partial_\psi$$

$$[h_{\eta^\Lambda,0}, h_{0,\tilde\eta_\Sigma}] = \delta^\Lambda_\Sigma k$$

$$\Lambda = (0,a) = 0,\dots,h^{1,1}$$

- **Shifts of the B-field**

$M_{\epsilon^a}\ :\ b^a \mapsto b^a + \epsilon^a$ supplemented by a change of RR-fields and NS-axion

Remarks

- redundancy

$$H_{\eta^0,0} \equiv e^{h_{\eta^0,0}} = T^{\eta^0}$$

Heisenberg shift → $H_{\eta^0,0}$; SL(2,ℝ) generator → T^{η^0}

- group structure

$$SL(2,\mathbb{R}) \ltimes N(\mathbb{R})$$

S-duality subgroup → $SL(2,\mathbb{R})$; nilpotent subgroup → $N(\mathbb{R})$

Quantum symmetries

Quantum corrections break the continuous isometries to *discreet* subgroups

$$SL(2,\mathbb{R}) \ltimes N(\mathbb{R}) \longrightarrow SL(2,\mathbb{Z}) \ltimes N(\mathbb{Z})$$

But it is *not* sufficient to replace real transformation parameters by integers!

Amendments: (S.A.,Persson,Pioline '10)

- $g \in SL(2,\mathbb{Z}) \;:\; \tilde{c}_a \mapsto \tilde{c}_a - c_{2,a}\varepsilon(g)$

  log of the multiplier system of Dedekind eta function

- $H_\Theta \;:\; \psi \mapsto \psi + \frac{1}{2}\langle \Theta, C\rangle + \sigma(\Theta)$

  quadratic refinement

$\Theta = (\eta^\Lambda, \tilde{\eta}_\Lambda)$

$\langle \Theta, \Theta' \rangle = \tilde{\eta}_\Lambda \eta'^\Lambda - \eta^\Lambda \tilde{\eta}'_\Lambda$

- $M_{\epsilon^a} \;:\; \psi \mapsto \psi + \frac{1}{6}\kappa_{abc}\epsilon^a c^b c^c + \kappa(M_{\epsilon^a})$

  character of symplectic group

Holomorphic prepotential:

$$F(X) = -\frac{\kappa_{abc}X^a X^b X^c}{6X^0} + \frac{1}{2}A_{\Lambda\Sigma}X^\Lambda X^\Sigma + \alpha'\text{-corr.}$$

$$A_{00} \in \mathbb{Z}$$
$$A_{0a} = \frac{c_{2,a}}{24} + \mathbb{Z}$$
$$A_{ab} = \frac{1}{2}\kappa_{abb} + \mathbb{Z}$$

↓ affects mirror map

$$\tilde{c}_\Lambda^{\rm IIB} = \tilde{c}_\Lambda^{\rm IIA} - A_{\Lambda\Sigma}c^\Sigma$$

↓ needs a correction

$$H_{\eta^0,0} \neq T^{\eta^0}$$

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  character of symplectic group

Quadratic refinement $s(\Theta) = e^{2\pi i \sigma(\Theta)}$ of the intersection form on the charge lattice

$$s(\Theta_1 + \Theta_2) = (-1)^{\langle \Theta_1, \Theta_2 \rangle} s(\Theta_1) s(\Theta_2)$$

- multiplies D-instantons

$$s(\gamma)\Omega(\gamma) e^{-2\pi |Z_\gamma|/g_s - 2\pi i \langle \gamma, C\rangle}$$

crucial for consistency with wall-crossing

- needed to cancel the anomaly in the NS-axion shift:

$$H_{\Theta_1} H_{\Theta_2} = H_{\Theta_1+\Theta_2} e^{\kappa \partial_\psi} \qquad \text{where}$$

$$\kappa = \frac{1}{2}\langle \Theta_1, \Theta_2 \rangle + \sigma(\Theta_1) + \sigma(\Theta_2) - \sigma(\Theta_1 + \Theta_2)$$

= integer due to the quadratic refinement

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$\Theta=(\eta^\Lambda,\tilde\eta_\Lambda)$

$\langle\Theta,\Theta'\rangle=\tilde\eta_\Lambda\eta'^\Lambda-\eta^\Lambda\tilde\eta'_\Lambda$

- $$M_{\epsilon^a}\ :\ \psi\mapsto\psi+\frac{1}{6}\kappa_{abc}\epsilon^a c^b c^c+\kappa(M_{\epsilon^a})$$

  character of symplectic group

$\mathcal{M}_{\rm HM}$ — S^1_ψ-bundle with curvature

$$F=\omega_{\mathcal{C}}+\frac{\chi_{\rm CY}}{24}\,\omega_{\mathcal{K}}$$

Kähler form on the torus of RR-fields

Kähler form for compl. Kähler moduli (1-loop effect)

ψ is not globally defined and the character allows to cancel fractional shifts due to monodromies in the Kähler moduli subspace

Since our group is abelian

$$\kappa(M_{\epsilon^a})=\kappa_a\epsilon^a$$

Anomaly

Let us consider

S-duality generator

$$S^{-1} H_{\eta^a} S \cdot \begin{pmatrix} b^a \\ c^a \\ \tilde{c}_a \\ \tilde{c}_0 \\ \psi \end{pmatrix} = M_{\eta^a} \cdot \begin{pmatrix} b^a \\ c^a \\ \tilde{c}_a \\ \tilde{c}_0 \\ \psi \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ A_{ab}\eta^b \\ \eta^a \left(\frac{c_{2,a}}{8} - \frac{1}{2} A_{ab}\eta^b \right) \\ \eta^a \left(\frac{c_{2,a}}{24} - \kappa_a \right) \end{pmatrix}$$

half-integer

generate shifts of

c^a b^a

transform as a doublet under SL(2,$\mathbb{Z}$)

Anomaly

the group law fails to hold

Resolution of the anomaly

The idea: the problem lies in the quadratic refinement

Defining equation

$$s(\Theta_1+\Theta_2)=(-1)^{\langle\Theta_1,\Theta_2\rangle}s(\Theta_1)s(\Theta_2)$$

General solution

$$\sigma(\Theta)=\frac{1}{2}\eta^\Lambda\tilde\eta_\Lambda+\eta^\Lambda\phi_\Lambda-\tilde\eta_\Lambda\theta^\Lambda$$

characteristics (generalized spin structure)

$s(\Theta)$ must be symplectic invariant

$$\rho=\begin{pmatrix}\mathcal{D} & \mathcal{C}\\ \mathcal{B} & \mathcal{A}\end{pmatrix}\ :\ \begin{pmatrix}\theta^\Lambda\\ \phi_\Lambda\end{pmatrix}\mapsto\rho\cdot\left[\begin{pmatrix}\theta^\Lambda\\ \phi_\Lambda\end{pmatrix}-\frac{1}{2}\begin{pmatrix}(\mathcal{A}^T\mathcal{C})_d\\ (\mathcal{D}^T\mathcal{B})_d\end{pmatrix}\right]$$

inhomogeneous transformation of characteristics

inconsistent with $M_{\eta^a}=S^{-1}H_{\eta^a}S$

do not affect characteristics

Observation: the characteristics and RR-fields always appear in combination

$$\begin{aligned} c^\Lambda-\theta^\Lambda &\mapsto c^\Lambda\\ \tilde c_\Lambda-\phi_\Lambda &\mapsto \tilde c_\Lambda\end{aligned}$$

redefinition

The redefined fields transform *inhomogeneously* under symplectic group

$$\begin{pmatrix}0\\0\\A_{ab}\epsilon^b\\ \frac{c_{2,a}}{8}\epsilon^a-\frac{1}{2}A_{ab}\epsilon^a\epsilon^b\end{pmatrix}$$

inhomogeneous term for M_{ϵ^a}

All anomalies cancel!

provided $\kappa_a=\dfrac{c_{2,a}}{24}$

Coclusions

- We showed that the consistent implementation of discrete isometries on the HM moduli space of Calabi-Yau compactifications requires a modification of the *monodromy* transformations of the RR-fields by *inhomogeneous* terms determined by the second Chern class and intersection numbers.

- After this modification, it was shown that NS5-brane instantons derived using S-duality are also consistent with the Heisenberg and monodromy invariance.

- The anomalous transformation of the RR-fields provides a generalization of the Freed-Witten anomaly.

Can one derive the anomalous terms from the Pfaffian of the Dirac operator?