The Theory of Resurgence in Matrix Models and Topological Strings

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Based on:
1308.1695, 1407.4821 with J. D. Edelstein (USC), R. Schiappa (IST), M. Vonk (UvA),
1501.01007 with R. Schiappa, R. Vaz (YITP),
work in progress with M. Mariño (UniGe), R. Schiappa

Outline

1. **Resurgence** in 10 minutes or less
2. The laboratory of **Matrix Models**
3. Making the most of **Topological Strings** using Resurgence

Resurgence

Resurgence was born independently

- Physics: Bender–Wu, Zinn-Justin, Voros, ...
- Mathematics: Écalle, Pham, ...

What to do with nonconvergent, asymptotic series?

$$\sum_{g=0}^{\infty} F_g \, \lambda^g, \qquad F_g \sim c \, A^{-g} \, g! \quad \text{as } g \to \infty.$$

- Physics: Number of Feynman diagrams grows factorially.
- Mathematics: Gevrey-1 series.

Origins of resurgence from physics

Bender–Wu, Anharmonic oscillator II a study of pertubation theory in large order, 1973

The quantum anharmonic oscillator is described by the time-independent Schrödinger equation

$$\left[\frac{d^2}{dx^2}+\frac{x^2}{4}+\frac{\lambda x^4}{4}-E(\lambda)\right]\psi(x)=0 \qquad (1.1)$$

$$E^K(\lambda)\sim K+\tfrac{1}{2}+\sum_{n=1}^{\infty}A_n^K\lambda^n\,. \qquad (1.3)$$

two equations. For large n,

$$A_n^K=\frac{(-1)^{n+1}12^K\sqrt{6}}{K!\,\pi^{3/2}}\,3^n\Gamma(n+K+\tfrac{1}{2})$$

$$\times\left[1-\frac{1}{n}\left(\frac{95}{72}+\frac{29K}{12}+\frac{17K^2}{12}\right)+O(n^{-2})\right] \qquad (1.8)$$

Origins of resurgence from mathematics

Écalle, Les fonctions resurgentes, Vol I, 1981

Signalons que les fonctions résurgentes les plus intéressantes, tant en soi que du point de vue des applications, sont celles qui sont liées simplement à leurs dérivées étrangères :

(I.8) $$\Delta_\omega \varphi = \text{fonction simple de } \omega \text{ et de } \varphi \qquad (\forall \omega \in \Omega)$$

De telles équations, dites <u>équations de résurgence</u>, expriment que la fonction φ "resurgit", quelque peu modifiée, en ses singularités. Ces φ là sont les fonctions résurgentes par excellence. Elles ont donné leur nom à toute la famille.

(iii) As for applications outside mathematics, at present there are none. Is this state of affairs likely ever to change ? It appears rather doubtful. For one thing, one is at a loss to say precisely which features

Origins of resurgence from mathematics

Écalle, Les fonctions resurgentes, Vol I, 1981

Addition :

As this text was going to press, Mr. A. Voros has drawn our attention to a most interesting source of resurgent functions, which may be added to the five sources listed on page 5 . These new functions are connected with the partition function $\mathrm{tr}\exp(-zH)$ of the hamiltonian $H = -d^2/dq^2 + V(q)$. In particular, the quartic oscillator, for $V(q) = q^4$, gives rise to a function verifying resurgence equations of the type :

$$\Delta_{m+ni}\,\varphi(z) = -A_{m+ni}\,\exp\left(-m\varphi(z) - n\varphi(iz)\right)$$

with constant A_ω and with $\omega = m+ni$ running through $\Omega = \mathbb{Z} + i\mathbb{Z}$.

Resurgence

1 Functions are represented by transseries

$$F(x) = \underbrace{\sum_{g=0}^{\infty} x^{-g}\, F_g^{(0)}}_{\text{divergent}} + \sum_{n\geq 1} \sigma^n\, \mathrm{e}^{-nAx} \underbrace{\sum_{g=0}^{\infty} x^{-g}\, F_g^{(n)}}_{\text{divergent}}$$

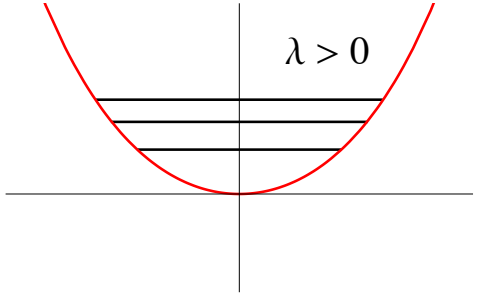
2 Perturbation theory knows about nonperturbative effects

$$F_g^{(0)} \longleftrightarrow A,\ F_h^{(n)}$$

An example: quartic integral

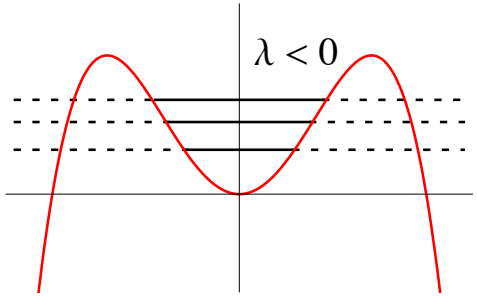
$$Z = \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi}}\, \mathrm{e}^{-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4}$$

$$F = \log Z = -\frac{1}{8}\lambda + \frac{1}{12}\lambda^2 - \frac{11}{96}\lambda^3 + \dots$$



Zero radius of convergence of perturbation theory

$$F_g \sim c\, g! \left(-\frac{3}{2}\right)^{-g}, \quad \text{as } g \to \infty.$$



Non-trivial saddle points for the potential:

$$V'(x_*) = 0 \Leftrightarrow x_* = 0,\ \pm i\sqrt{\frac{6}{\lambda}}, \qquad V\left(\pm i\sqrt{\frac{6}{\lambda}}\right) = -\frac{3}{2}\frac{1}{\lambda}$$

[Berry, Howls]

Resurgence: Borel transform and alien derivative

Borel transform,

$$F^{(n)}(x) = \underbrace{\sum_{g=0} F_g^{(n)}\, x^{-g-1}}_{\text{divergent}} \longmapsto \mathcal{B}[F^{(n)}](\xi) = \underbrace{\sum_{g=0}^{\infty} \frac{F_g^{(n)}}{g!}\, \xi^g}_{\text{convergent}} .$$

$F^{(n)}(x)$ is **resurgent** if $\mathcal{B}[F^{(n)}](\xi)$ has a discrete set of singularities and analytic continuations for any path around them.

$F^{(n)}(x)$ is **simply resurgent** if around each singularity ω,

$$\mathcal{B}[F^{(n)}](\xi) = \frac{\boxed{a_\omega}}{2\pi i(\xi-\omega)} + \mathcal{B}[\boxed{G_\omega}](\xi-\omega)\frac{\log(\xi-\omega)}{2\pi i} + \text{hol. func.}$$

Alien derivative Δ_ω,

$$\Delta_\omega F^{(n)}(x) := \boxed{a_\omega} + \boxed{G_\omega(x)}, \qquad [\mathrm{e}^{-\omega\, x}\Delta_\omega, \partial] = 0.$$

Resurgence: $G_\omega(x)$ is one of $F^{(m)}(x)$ for some m.

Resurgence: Resummation and large-order growth

Resummation in x,

$$F^{(n)}(x) \longmapsto \mathcal{B}[F^{(n)}](\xi) \longmapsto \mathcal{S}F^{(n)}(x) := \int_0^\infty d\xi\, \mathrm{e}^{-x\,\xi}\, \mathcal{B}[F^{(n)}](\xi).$$

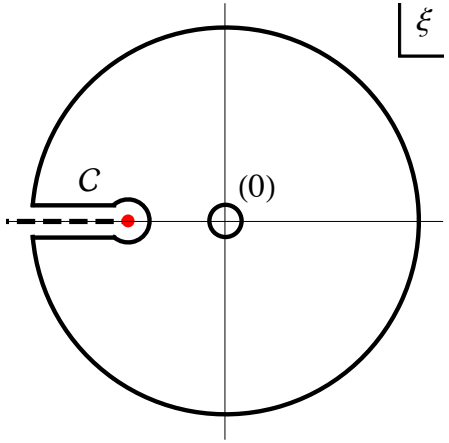
If there are poles ω along integration contour,

$$\mathcal{S}_+ \neq \mathcal{S}_-, \qquad \mathcal{S}_+ = \mathcal{S}_- \circ \mathfrak{S}, \qquad \mathfrak{S} = \exp\left(\sum_{\omega\in\mathbb{R}^+} \mathrm{e}^{-\omega x}\, \Delta_\omega\right).$$

Stokes automorphism

Large-order growth ($g \to \infty$),

$$F_g^{(0)} = \frac{g!}{2\pi i} \oint_{(0)} d\xi\, \xi^{-g-1} \mathcal{B}[F^{(0)}](\xi)$$
$$= \frac{g!}{2\pi i} \oint_{\mathcal{C}} d\xi\, \xi^{-g-1} \mathcal{B}[F^{(0)}](\xi)$$



Resurgence: Resummation and large-order growth

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Stokes automorphism

Large-order growth $(g \to \infty)$,

for functions satisfying differential equation, difference equation,

Stokes constant

$$[\mathrm{e}^{-\omega\, x}\Delta_\omega, \partial] = 0 \quad \oplus \quad \omega = nA \quad \oplus \quad (\text{Eq. for } F) \quad \Rightarrow \quad \Delta_{nA} F^{(0)} = S_1\, n\, F^{(n)}$$

bridge equation

$$\boxed{F_g^{(0)} \sim \sum_{n=1}^{\infty} \frac{S_1^n}{2\pi i} \sum_{h=0}^{\infty} \frac{\Gamma(g-h)}{(nA)^{g-h}}\, F_h^{(n)}, \quad \text{as } g \to \infty.}$$

Quartic Matrix Model

Quartic matrix integral

$$Z(N,t) = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{dx_i}{2\pi} \, \Delta(x)^2 \, \mathrm{e}^{-\frac{t}{N} \sum_i V(x_i)},$$

$$V(x) = \frac{1}{2} x^2 + \frac{\lambda}{4!} x^4, \quad \Delta(x)^2 = \prod_{i<j} (x_i - x_j)^2, \quad t = g_{\mathrm{YM}}^2 N = g_s N.$$

$Z(N,t)$ can be **computed exactly** for any value $N \in \mathbb{N}$.

$F = \log Z$ can be **computed perturbatively** as a $1/N^2$ 't Hooft expansion

$$F \sim F^{(0)} = \sum_{g=0}^{\infty} N^{2-2g} \, F_g^{(0)}(t), \quad \text{as } N \to \infty,$$

[Bessis, Itzykson, Zuber]

and **beyond perturbation** theory using orthogonal polynomials, [Mariño]

[Aniceto, Schiappa, Vonk]

$$F = \sum_{g=0}^{\infty} N^{2-2g} \, F_g^{(0|0)}(t) + \sum_{n,m} \sigma_1^n \sigma_2^m \, \mathrm{e}^{-(n-m)A(t)N} \sum_k \log^k(N) \sum_{g=0}^{\infty} N^{-g} F_h^{(n|m)[k]}(t)$$

Quartic Matrix Model

Resummation:

$$\text{exact } F \quad = \quad \text{resummation}(\text{transseries } F)|_{N\in\mathbb{N}}$$

Stokes phenomenon:

- $\sigma_1 = \sigma_2 = 0$: Perturbation theory is enough.
- $\sigma_1 = S_1 \neq 0$, $\sigma_2 = 0$: Instanton corrections.

$$\mathcal{S}_+ = \mathcal{S}_- \circ \mathfrak{S}, \qquad \mathfrak{S} = \exp\left(\sum_{\omega\in\mathbb{R}^+} \mathrm{e}^{-\omega x}\,\Delta_\omega\right)$$

$$\mathrm{e}^{-nAx}\,\Delta_{nA} \simeq S_1\,\partial_{\sigma_1}, \qquad \boxed{\mathcal{S}_+ F(N;\sigma_1) = \mathcal{S}_- F(N;\sigma_1 + S_1)}$$

Transseries resummation for $N = 3$ and $t = 6$

Instanton corrections are important ($\sigma_1 = S_1$):

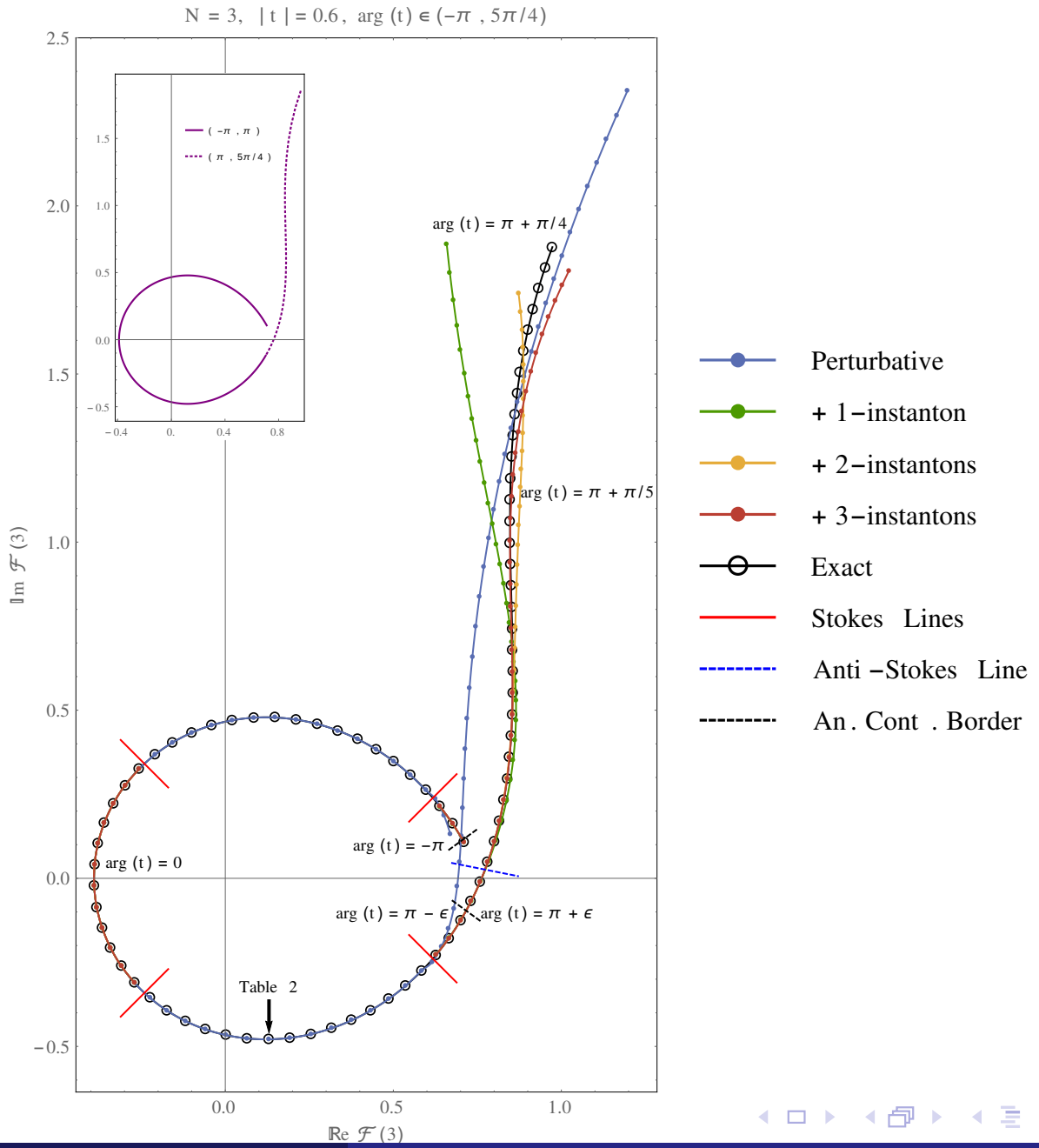
Sector	$S_0 F^{(n)}$
Perturbative	$-1.973\,899\,279\,493\,161\,74\ldots$
1-Instanton	$-0.000\,020\,359\,080\,917\,15\ldots$
2-Instanton	$-0.000\,000\,000\,300\,789\,88\ldots$
3-Instanton	$-0.000\,000\,000\,000\,004\,71\ldots$
Total	$-1.973\,919\,638\,874\,873\,50\ldots$
Exact	$-1.973\,919\,638\,874\,873\,50\ldots$

Resummation of F for $N = 3$ and $t = 0.6\mathrm{e}^{2\pi i/3}$

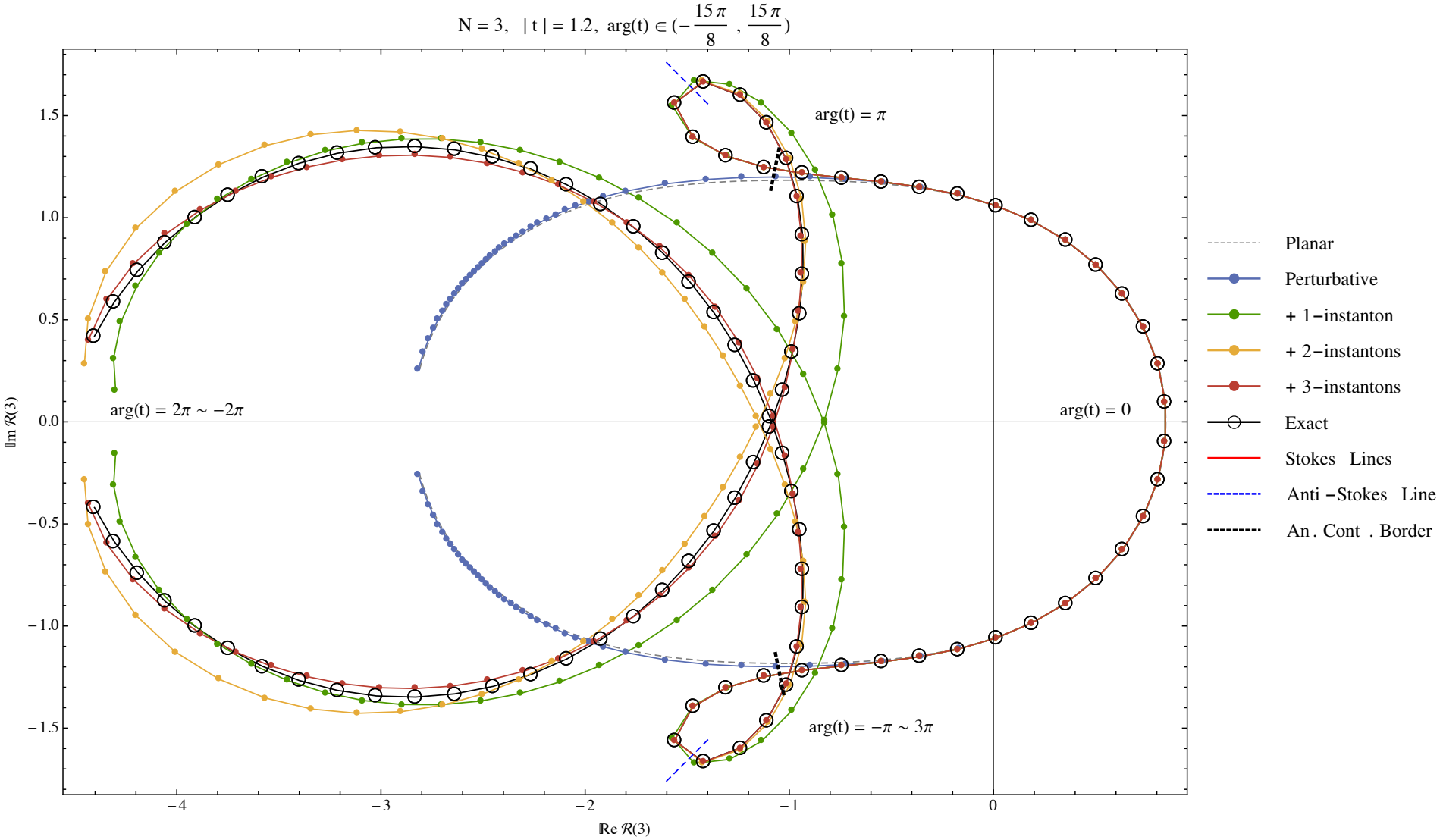
Perturbation theory is enough ($\sigma_1 = 0$):

Sector	Re $S_0 F^{(n)}$	Im $S_0 F^{(n)}$
Perturbative	$+0.130\,991\,945\,237\,228\ldots$	$-0.478\,840\,360\,187\,836\ldots$
~~1-Instanton~~	~~$-0.000\,070\,474\,759\,944\ldots$~~	~~$-0.000\,010\,860\,987\,563\ldots$~~
~~2-Instanton~~	~~$-0.000\,000\,002\,360\,007\ldots$~~	~~$-0.000\,000\,001\,378\,327\ldots$~~
~~3-Instanton~~	~~$-0.000\,000\,000\,000\,097\ldots$~~	~~$-0.000\,000\,000\,000\,095\ldots$~~
Exact	$+0.130\,991\,945\,237\,228\ \ldots$	$-0.478\,840\,360\,187\,836\ldots$

Resummation of F for $N = 3$ and $|t| = 0.6$



Resummation of R for $N = 3$ and $|t| = 1.2$



Interlude

$$
\begin{tikzcd}
\text{Perturbative } F_g^{(0)} \arrow[rr, "\text{large-order } (g \to \infty)"] \arrow[d, "\text{recursion}"'] \arrow[drr, dashed, "\text{LO}"] \arrow[ddr, dashed, "\text{E}"'] & & \text{Resurgence Relations} \arrow[d, "\text{reconstruction}"] \\
\text{Equation} \arrow[rr, "\text{formal solution}", "+\text{ extra info}"'] & & \text{Transseries } F \arrow[r] \arrow[d, "\text{resummation}"'] & F_g^{(n)} \arrow[uu, bend right, "\text{large-order}"'] \\
\text{Other approaches to } F \arrow[rr, leftrightarrow, "\text{compatibility}"] & & \text{Physical } F
\end{tikzcd}
$$

Topological String Theory

Topological string theory has **no nonperturbative definition** from first principles.

The **resurgent approach** is viable:

[Bershadsky, Cecotti, Ooguri, Vafa]

- Can compute many $F_g^{(0)}$'s [holomorphic anomaly equations]
- Can provide an equation [extended holomorphic anomaly equations.]
- Can compare with other approaches [refined string and spectral theory]

Topological string theory has many faces and relations to other fields in pure mathematics and theoretical physics:

- Mirror symmetry.
- Enumerative geometry.
- String theory.
- Gauge theories and matrix models.
- Modular symmetry and generalizations (see E. Scheidegger's talk.)

Resurgent approach to Topological String Theory (e.g. local $\mathbb{P}^2$)

$$
\begin{tikzcd}
\text{Perturbative } F_g^{(0)} \arrow[r, "\text{large-order } (g \to \infty)"] \arrow[d, "\text{recursion}"'] \arrow[dr, dashed, "\text{LO}"] \arrow[ddr, dashed, "\text{HAE}"'] & \text{Resurgence Relations} \arrow[d, "\text{reconstruction}"] & \\
\text{Equation} \arrow[r, "\text{formal solution}", "+\text{ extra info}"'] & \text{Transseries } F \arrow[r] \arrow[d, "\text{resummation}"'] & F_g^{(n)} \arrow[uu, bend right, "\text{large-orde}"'] \\
\text{Other approaches to } F \arrow[r, leftrightarrow, "\text{compatibility}"] & \text{Physical } F &
\end{tikzcd}
$$

Resurgent approach to Topological String Theory (e.g. local $\mathbb{P}^2$)

$$\partial_S F_g^{(0)} = \partial_z^2 F_{g-1}^{(0)} + \sum_{h=1}^{g-1} \partial_z F_h^{(0)} \, \partial_z F_{g-h}^{(0)}$$

$$\sum_g g_s^{2g-2} F_g^{(0)}$$

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order ($g \to$

$$F_g^{(\mathbf{0})} \sim \sum_{\boldsymbol{n}} \sum_h \frac{(g-h)!}{(\boldsymbol{n}\cdot\boldsymbol{A})^{g-h}} F_h^{(\boldsymbol{n})}$$
$$\pm A_1, \pm A_2, \pm A_3, \pm A_{\mathrm{K}},$$
$$F_g^{(1\boldsymbol{e}_1)}, F_g^{(1\boldsymbol{e}_2)}, F_g^{(2\boldsymbol{e}_1)}, F_g^{(\boldsymbol{e}_{1,1})}$$

LO

HAE

recursion

reconstruction

large-orde

Equation $\longrightarrow$ Transseries F $\longrightarrow$ $F_g^{(n)}$

formal solution
+ extra info

resummation

Other approaches to F $\xleftrightarrow{\text{compatibility}}$ Physical F

Resurgent approach to Topological String Theory (e.g. local $\mathbb{P}^2$)

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$$\partial_S F = g_s^2(\partial_z^2 F + (\partial_z F)^2)$$

formal solution
+ extra info

Transseries F

$F_g^{(n)}$

resummation

Other approaches to F $\xleftrightarrow{\text{compatibility}}$ Physical F

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formal solution
+ extra info

$$F = \sum_{\boldsymbol{n}=\mathbf{0}}^{\infty} \boldsymbol{\sigma}^{\boldsymbol{n}} \mathrm{e}^{-\boldsymbol{n}\cdot\boldsymbol{A}/g_s} \sum_{g=0}^{\infty} g_s^g F_g^{(\boldsymbol{n})}$$

$\to F_g^{(n)}$

resummation

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Resurgent approach to Topological String Theory (e.g. local $\mathbb{P}^2$)

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formal solution

+ extra info

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resummation

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formal solution

+ extra info

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$$F_g^{(\boldsymbol{n})}$$

resummation

compatibility

Other approaches to F ← Physical F (holomorphic)

Resurgent approach to Topological String Theory (e.g. local $\mathbb{P}^2$)

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$$\partial_S F = g_s^2(\partial_z^2 F + (\partial_z F)^2)$$

formal solution
+ extra info

$$F = \sum_{\boldsymbol{n}=\mathbf{0}}^{\infty} \boldsymbol{\sigma}^{\boldsymbol{n}} \mathrm{e}^{-\boldsymbol{n}\cdot \boldsymbol{A}/g_s} \sum_{g=0}^{\infty} g_s^g F_g^{(\boldsymbol{n})}$$

$$F_g^{(\boldsymbol{n})}$$

resummation

compatibili

Other Exact F s to F

Physical F (holomorphic)

Nonperturbative definition of Topological String Theory

For toric Calabi–Yau geometries (e.g. local $\mathbb{P}^2$) conjecture

$$\text{holomorphic } F = \log Z.$$

[Hatsuda, Mariño, Moriyama, Okuyama]
{Grassi, Kallen, Kashaev, Zakany}

Construction for **local** $\mathbb{P}^2$:

complex structure

Underlying Riemann surface: $\mathrm{e}^x + \mathrm{e}^y + \mathrm{e}^{-x-y} + \psi = 0$

Operator: $\hat{\mathcal{O}} := \mathrm{e}^{\hat{x}} + \mathrm{e}^{\hat{y}} + \mathrm{e}^{-\hat{x}-\hat{y}}, \quad [\hat{x}, \hat{y}] = i\hbar, \quad \hat{\rho} := \hat{\mathcal{O}}^{-1}$

Finite traces: $Z_\ell(\hbar) := \operatorname{Tr} \hat{\rho}^\ell, \quad \ell = 1, 2, \ldots$

Partition functions: $\sum_{N=0}^{\infty} \boxed{Z(N,\hbar)} \kappa^N := \exp\left(- \sum_{\ell=1}^{\infty} Z_\ell(\hbar) \frac{(-\kappa)^\ell}{\ell} \right)$

Dictionary:

conifold flat coordinate

$$\hbar = \frac{4\pi^2}{g_s}, \qquad N = \frac{t_{\mathrm{c}}(\psi)}{g_s}.$$

Examples:

$$Z(1, 2\pi) = \frac{1}{9}, \qquad Z(2, 4\pi) = \boxed{\frac{5}{324} - \frac{1}{12\sqrt{3}\pi}}.$$

Resummation example: $N = 2$, $\hbar = 4\pi$ ($g_s = \pi$)

Exact result vs resummation of perturbation theory:

$$\text{exact } (-F) = 9.049\,862\,102\,738\,02\ldots = -\log\left(\frac{5}{324} - \frac{1}{12\sqrt{3}\pi}\right)$$

$$\text{pert } (-F^{(0)}) = 9.049\,862\,103\,051\,21\ldots$$

$$\Delta F = \boxed{3.131\,8\ldots \cdot 10^{-10}}$$

The difference is a nonperturbative effect, exponentially small, e^{-A/g_s}:

$$\text{1-inst } F^{(1)} = \boxed{3.131\,840\ldots \cdot 10^{-10}}$$

Relevant instanton actions are A_{K} and A_2.
Resummation is not well understood yet in all cases.

Same principles should apply to any other CY geometry, **local or compact**.