Monotonicity and Convexity for Discrete Fractional Difference Operators

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Introduction and Preliminaries

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For example,

$$\Delta^2 f(t) = \Delta[\Delta f(t)] = \Delta[f(t+1) - f(t)] = f(t+2) - 2f(t+1) + f(t).$$

In the continuous calculus, the function $f(t) = t^n$, $n \in \mathbb{R} \setminus \{0\}$, plays a special role because it is so easy to find its derivative – namely, $f'(t) = nt^{n-1}$.

But $\Delta [t^n]$ is not as simple as $\frac{d}{dt} [t^n]$.

So, what function might take the place of $f(t) = t^n$ in the difference calculus? It is the **falling factorial function**.

Definition

We define

$$t^{\underline{\nu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)},$$

for any real numbers t and ν for which the right-hand side is defined. We also appeal to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}} := 0$.

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Let's look at an example.

Example

Let $f(t) = t^{\underline{\frac{3}{2}}}$. Then it holds that

$$\Delta[f(t)] = \frac{3}{2}t^{\underline{\frac{1}{2}}} = \frac{3}{2}\cdot\frac{\Gamma(t+1)}{\Gamma\left(t+1-\frac{1}{2}\right)} = \frac{3}{2}\frac{\Gamma(t+1)}{\Gamma\left(t+\frac{1}{2}\right)}.$$

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Consider now instead the ***fractional*** forward difference (or fractional Δ-derivative) of order ν defined by

$$\Delta_a^{\nu} f(t) := \frac{1}{\Gamma(-\nu)} \sum_{s=a}^{t+\nu} (t-s-1)^{\underline{-\nu-1}} f(s),$$

where $\nu \in (N-1, N)$, for $N \in \mathbb{N}$, and $t \in \{a+N-\nu, a+N-\nu+1, \ldots\}$.

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Note: Notice that this definition incorporates a nonlocality.

Note: The operator Δ_a^ν induces a domain shift!

So, why should we care at all about these generalizations. (Are they purely "academic"?) Here are a few possible answers:

1. Analysis of fractional equations (both discrete and continuous) often is more subtle and complicated than the integer-order counterparts.
2. Fractional equations have a memory property.
3. Fractional equations have built-in nonlocalities.
4. Fractional-order equations can provide for better models, in certain cases, than integer-order equations.
5. Certain classical topics in the theory of ODEs and difference equations seem to be exceptionally complicated and mathematically interesting in the fractional setting.
6. Fractional integrals and derivatives have proved to be useful in regularity theory and the analysis of PDEs.

Monotonicity Result

We begin with the following result, which was originally deduced by Dahal and Goodrich (2014) and then refined by Baoguo, Erbe, and Peterson (2015).

Theorem

Let $y : \mathbb{N}_a \to \mathbb{R}$ be a nonnegative function satisfying $\Delta y(a) \geq 0$. Fix $\nu \in (1,2)$ and suppose that $\Delta_a^{\nu} y(t) \geq 0$ for each $t \in \mathbb{N}_{a+2-\nu}$. Then y is increasing on $\mathbb{N}_a$.

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Notation: We write $\mathbb{N}_a := \{a, a+1, a+2, \ldots\}$ for any $a \in \mathbb{R}$.

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Note: One must require *some* relationship between $y(a)$ and $y(a+1)$.

Example

Let $f : \mathbb{N}_0 \to \mathbb{R}$ be defined by $f(t) = 2^{-t}$, and assume that $\frac{2+\sqrt{2}}{2} < \nu < 2$. It can be shown that

- $\Delta_0^\nu f(t) \geq 0$ for $t \in \mathbb{N}_{2-\nu}$; and
- $f(t) \geq 0$ for $t \in \mathbb{N}_0$,

but $f(t)$ is **NOT** increasing on $\mathbb{N}_1$. The problem is that $\Delta f(0) < 0$.

One can further refine the preceding monotonicity theorem in the following way. This result is due to Baoguo, Erbe, Goodrich, and Peterson (2015).

Theorem

Assume that $f : \mathbb{N}_a \to \mathbb{R}$ *and that* $\Delta_a^\nu f(t) \geq 0$, *for each* $t \in \mathbb{N}_{a+2-\nu}$, *with* $1 < \nu < 2$. *If*

$$f(a+1) \geq \frac{\nu}{k+2} f(a)$$

for each $k \in \mathbb{N}_0$, *then* $\Delta f(t) \geq 0$, *for* $t \in \mathbb{N}_{a+1}$.

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Theorem

Assume that $f : \mathbb{N}_a \to \mathbb{R}$ *and that* $\Delta_a^\nu f(t) \geq 0$*, for each* $t \in \mathbb{N}_{a+2-\nu}$*, with* $1 < \nu < 2$*. If*

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for each $k \in \mathbb{N}_0$*, then* $\Delta f(t) \geq 0$*, for* $t \in \mathbb{N}_{a+1}$*.*

Note: This result does NOT require that f be increasing "at" $t = a$. Thus, we are able to weaken one of the the assumptions in the original result.

As an application we have the following.

Example

Assume that $f : \mathbb{N}_a \to \mathbb{R}$ is a map satisfying the following.

- $\Delta_a^{\nu} f(t) \geq 0$, for $t \in \mathbb{N}_{a+2-\nu}$
- $f(a) \geq 0$
- $f(a+1) \geq \frac{\nu}{2} f(a)$

Then f is increasing on $\mathbb{N}_{a+1}$.

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Then f is increasing on $\mathbb{N}_{a+1}$.

Note: The point is that if $f(a) \geq 0$, then

$$f(a+1) \geq \frac{\nu}{2} f(a) \geq \frac{\nu}{3} f(a) \geq \frac{\nu}{4} f(a) \geq \cdots \geq 0$$

holds. So, the condition $f(a+1) \geq \frac{\nu}{k+2} f(a)$ immediately holds for all $k \in \mathbb{N}_0$.

Convexity Results

We begin with the following result, which was originally discovered by Goodrich (2014) and then refined by Baoguo, Erbe, and Peterson (2015).

Theorem

If $2 < \mu < 3$ *and* $y : \mathbb{N}_0 \to \mathbb{R}$ *satisfies*

1. $y(0) = 0$;
2. $\Delta y(0) \geq 0$;
3. $\Delta^2 y(0) \geq 0$; *and*
4. $\Delta_0^{\mu} y(t) \geq 0$, *for each* $t \in \mathbb{N}_{3-\mu}$,

then $\Delta^2 y(t) \geq 0$, *for each* $t \in \mathbb{N}_0$.

We can provide an application of this result to a problem in discrete fractional-order boundary value problems.

Example

If the continuous function $f : \mathbb{N}_{\mu-1}^{\mu+b} \times \mathbb{R} \to \mathbb{R}$ is nonnegative and $2 < \mu < 3$, then, for $t \in \mathbb{N}_0^{b+1} =: \{0, 1, \ldots, b+1\}$, the boundary value problem

$$\Delta_{\mu-3}^{\mu} y(t) = f(t+\mu-1, y(t+\mu-1))$$

$$y(\mu-3) = 0 = \Delta y(\mu-3)$$

$$y(\mu+b+1) = 0$$

has no nontrivial positive solution.

Similar to the monotonicity result, the preceding convexity-type result has recently been refined in the following way by Goodrich (2015).

Theorem

Fix $\nu \in (2,3)$ *and suppose that* $\Delta_a^\nu f(t) \geq 0$ *for each* $t \in \mathbb{N}_{3+a-\nu}$. *If for each* $k \in \mathbb{N}_{-1}$ *it holds that*

$$\frac{1}{-\nu+1} f(a+2) + \frac{\nu+2+k}{(\nu-1)(3+k)} f(a+1) - \frac{\nu}{(3+k)(4+k)} f(a) \leq 0,$$

then $\Delta^2 f(t) \geq 0$ *for each* $t \in \mathbb{N}_{a+1}$.

Importantly, we make the following observation regarding the hypothesis

$$\frac{1}{-\nu+1}f(a+2)+\frac{\nu+2+k}{(\nu-1)(3+k)}f(a+1)-\frac{\nu}{(3+k)(4+k)}f(a)\leq 0,$$

which occurs in the preceding “improved convexity” result.

Example

If we put $f(a)=0$, $f(a+1)=1$, and $f(a+2)=1.9$ and we also fix $\nu=\frac{5}{2}\in(2,3)$, then we calculate, for example,

$$\frac{1}{-\nu+1}f(a+2)+\frac{\nu+1}{2(\nu-1)}f(a+1)-\frac{\nu}{6}f(a)\leq 0;$$

the condition holds for all $k\in\mathbb{N}_{-1}$, in fact.

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the condition holds for all $k\in\mathbb{N}_{-1}$, in fact. **Yet we observe that $\Delta^2 f(a)=-\frac{1}{10}<0$.**

Selected References

[1] J. Baoguo, L. Erbe, A. Peterson, *Two monotonicity results for nabla and delta fractional differences*, Arch. Math. (Basel) **104** (2015), 589–597

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Thank you for your attention!