Weyl theory for Dirac operators and applications to initial-boundary value problems for integrable wave equations

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Contents

Weyl theory is an important tool in initial-boundary value problems and, vice versa,
initial-boundary value problems are of interest in Weyl theory.

We start with the well-known zero curvature equation

$$G_t(x,t,z) - F_x(x,t,z) + [G(x,t,z), F(x,t,z)] = 0 \qquad (1)$$

$$\Big([G,F] := GF - FG, \qquad G_t := \frac{\partial}{\partial t} G\Big),$$

which is the *compatibility condition* of the auxiliary linear systems

$$w_x = Gw, \qquad w_t = Fw, \qquad (2)$$

where G and F are $m \times m$ matrix functions, and z is the spectral parameter.

Equation (1) is an equivalent of Lax's equation and as such it is actively used for representation of various integrable equations.

The term "compatibility condition" is very often used with respect to (1) and it is easy to derive (1) from (2).

In other words, it is easy to show that

$$G_t - F_x + [G, F] = 0 \tag{3}$$

is a necessary compatibility condition for the systems

$$w_x = Gw, \quad w_t = Fw$$

but the proof of sufficiency is more complicated.

More precisely, assume that (3) holds in the semistrip

$$\Omega_a = \{(x,\ t):\ 0 \le x < \infty,\ 0 \le t < a\}. \tag{4}$$

Is the system $w_x = Gw$, $w_t = Fw$ compatible in Ω_a?

Initial-boundary conditions

$$w(x, 0, z) = W(x, 0, z), \quad w(0, t, z) = R(0, t, z), \tag{5}$$

follow from the normalization $w(0, 0, z) = I_m$ and def-s of W, R:

$$W_x(x, t, z) = G(x, t, z)W(x, t, z), \quad W(0, t, z) = I_m;$$
$$R_t(x, t, z) = F(x, t, z)R(x, t, z), \quad R(x, 0, z) = I_m.$$

Thus, we have an initial-boundary value problem

$$w_x = Gw, \quad w_t = Fw; \tag{6}$$

$$w(x,0,z) = W(x,0,z), \quad w(0,t,z) = R(0,t,z). \tag{7}$$

Recall that $W_x(x,t,z) = G(x,t,z)W(x,t,z)$,
$R_t(x,t,z) = F(x,t,z)R(x,t,z)$.

Theorem 1. Let $m \times m$ matrix functions G and F and their derivatives G_t and F_x exist on the semi-strip Ω_a, let G, G_t, and F be continuous with respect to x and t on Ω_a, and let $G_t - F_x + [G,F] = 0$. Then we have the equality

$$W(x,t,z)R(0,t,z) = R(x,t,z)W(x,0,z). \tag{8}$$

Hence, $G_t - F_x + [G,F] = 0$ implies (8) and so the solution of (6), (7) exists and is given by the formula

$$w(x,t,z) = W(x,t,z)R(0,t,z) = R(x,t,z)W(x,0,z).$$

The first proof of "sufficiency" for the "compatibility condition" and for the factorization f-la

$$W(x,t,z)R(0,t,z) = R(x,t,z)W(x,0,z) \qquad (9)$$

were given in L.A. Sakhnovich, St. Petersburg Math. J. **5**:1, 1994 and in greater detail under weaker conditions in
ALS, J. Diff. Eq-s **252**, 2012.

The initial-boundary value problem

$$\begin{aligned} &w_x = Gw, \quad w_t = Fw; \\ &w(x,0,z) = W(x,0,z), \quad w(0,t,z) = R(0,t,z) \end{aligned}$$

or, equivalently, the factorization formula (9) is basic in the derivation of the evolution of the Weyl function of the system $W_x = GW$.

On the other hand, the evolution of the Weyl function plays an essential role in the study of the initial-boundary value problems for integrable systems which admit representation
$G_t - F_x + [G,F] = 0$.

Now, recall that for the case

$$G = \mathrm{i}(zj + jV), \quad F = -\mathrm{i}\big(z^2 j + zjV - \big(\mathrm{i}V_x - jV^2\big)/2\big), \tag{10}$$

$$j = \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix}, \quad V = \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}, \quad m_1 + m_2 = m, \tag{11}$$

where I_{m_1} is the $m_1 \times m_1$ identity matrix and v is an $m_1 \times m_2$ matrix function,
$G_t - F_x + [G, F] = 0$ is equivalent to the famous defocusing nonlinear Schrödinger eq-n (dNLS)

$$2v_t = \mathrm{i}(v_{xx} - 2vv^* v). \tag{12}$$

Here the system

$$W_x = GW = \mathrm{i}\big(zj + jV(x)\big) W(x, z)$$

is a selfadjoint Dirac (also called ZS-AKNS) system.
Let us consider the case of dNLS as an example.

Consider selfadjoint Dirac system $W_x = GW$, $G = \mathrm{i}(zj + jV)$, where V is locally summable on $[0, \infty)$. Let $W(0, z) = I_m$.

Then Weyl function is an $m_2 \times m_1$ holomorphic matrix function, which satisfies the inequality

$$\int_0^\infty \begin{bmatrix} I_{m_1} & \varphi(z)^* \end{bmatrix} W(x,z)^* W(x,z) \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix} dx < \infty, \quad z \in \mathbb{C}_+.$$

The Weyl function always exists and it is unique.

The inverse problems to recover V from the scalar or square, spectral or Weyl matrix functions is closely connected with M.G. Krein, B.L. Levitan, V.A. Marchenko, and (in the context of Borg-Marchenko theorems) with F. Gesztesy and coauthors.

Dirac system $W_x = GW$ (where V is locally square summable) is uniquely recovered from the Weyl function φ.

We shall present a procedure to recover V in the second half of the talk, see more details in ALS, arXiv:1401.3605.

See further results and references in ALS, L.A. Sakhnovich, I.Ya. Roitberg "Inverse Problems ...", de Gruyter, 2013.

The next formula gives an important property of Weyl functions:

$$\varphi(z) = \lim_{b\to\infty} \varphi_b(z) \tag{13}$$

for any set of functions $\varphi_b(z) \in \mathcal{N}(b,z)$. Here $\mathcal{N}(b,z)$ is the set (Weyl circle) of functions of the form

$$\begin{aligned}\varphi(b,z,\mathcal{P}) = \begin{bmatrix} 0 & I_{m_2} \end{bmatrix} W(b,z)^{-1}\mathcal{P}(z) \\ \times \Big(\begin{bmatrix} I_{m_1} & 0 \end{bmatrix} W(b,z)^{-1}\mathcal{P}(z)\Big)^{-1},\end{aligned} \tag{14}$$

where $\mathcal{P}(z)$ are $m \times m_1$ nonsingular meromorphic matrix functions with property-j, i.e.,

$$\mathcal{P}(z)^*\mathcal{P}(z) > 0, \quad \mathcal{P}(z)^* j \mathcal{P}(z) \geq 0 \quad (z \in \mathbb{C}_+). \tag{15}$$

In order to derive evolution $\varphi(t,z)$, we insert an additional variable t into functions φ and W and substitute the equality $W(b,t,z)^{-1} = R(0,t,z)W(b,0,z)^{-1}R(b,t,z)^{-1}$ into (14) .

Passing (on the previous frame) to the limit we obtain:

Theorem 2 (ALS). Let an $m_1 \times m_2$ matrix function $v(x,t)$ be continuously differentiable on the semistrip Ω_a and let v_{xx} exist. Assume that v satisfies the dNLS equation

$$2v_t = \mathrm{i}(v_{xx} - 2vv^*v)$$

as well as the following inequalities (for all $0 \le t < a$ and some values $M(t) \in \mathbb{R}_+$) :

$$\sup_{x \in \mathbb{R}_+,\, 0 \le s \le t} \|v(x,s)\| \le M(t). \tag{16}$$

Then the evolution $\varphi(t,z)$ of the Weyl functions of Dirac systems $W_x(x,t,z) = G(x,t,z)W(x,t,z)$ is given (for $z \in \mathbb{C}_+$) by the equality

$$\begin{aligned}\varphi(t,z) =& \big(R_{21}(t,z) + R_{22}(t,z)\varphi(0,z)\big) \\ & \times \big(R_{11}(t,z) + R_{12}(t,z)\varphi(0,z)\big)^{-1}.\end{aligned} \tag{17}$$

The evolution formula

$$\varphi(t,z) = \big(R_{21}(t,z) + R_{22}(t,z)\varphi(0,z)\big) \times \big(R_{11}(t,z) + R_{12}(t,z)\varphi(0,z)\big)^{-1}. \tag{18}$$

proves important in several problems, including uniqueness problems and blow up solutions.

We note that $\varphi(0,z)$ in (18) is determined by the initial condition $v(x,0)$, that $R(t,z)$ is determined by the boundary conditions $v(0,t)$ and $v_x(0,t)$ via equation $R_t(t,z) = F(0,t,z)R(t,z)$ and that $v(x,t)$ can be uniquely recovered from $\varphi(t,z)$.

The main difficulty is that in various approaches to these problems the necessary initial-boundary conditions make our wave equation overdetermined. Thus, the reduction of the initial-boundary conditions is crucial.

Next, we go to the approach suggested in the papers ALS, arXiv:1405.3500 and ALS, J. Math. Anal. Appl. **423** (2015)

Consider dNLS with quasi-analytic boundary conditions and smoothness near zero in the semistrip Ω_a

By $C_\varepsilon(\Omega)$ we denote the class of $m_1 \times m_2$ matrix functions $v(x,t)$, which are continuously differentiable and are such that v_{xx} exists on Ω_a. Moreover, it is required that for each k there is a value $\varepsilon_k = \varepsilon_k(v) > 0$ such that v is k times continuously differentiable with respect to x in the square

$$\Omega(\varepsilon_k) = \{(x,\ t) : \quad 0 \le x \le \varepsilon_k, \quad 0 \le t \le \varepsilon_k\}, \quad \Omega(\varepsilon_k) \subset \Omega_a.$$

Proposition 3. Assume that $v \in C_\varepsilon(\Omega)$ satisfies dNLS on Ω_a. Then, for each integer $r \ge 0$ and values $0 \le k \le r$, the functions $\left(\frac{\partial^k}{\partial t^k} v\right)(x,0)$ and $\left(\frac{\partial^k}{\partial t^k} v_x\right)(x,0)$ may be uniquely recovered (on the interval $0 \le x \le \varepsilon_{4r}$) from the initial condition $v(x,0) = h(x)$.

Thus, we derive our next theorem.

Theorem 4. Let conditions of Proposition 3 and the inequalities

$$\sup_{x\in\mathbb{R}_+,\, 0\le s\le t} \|v(x,s)\| \le M(t)$$

hold. Assume that $v(0,t)$ and $v_x(0,t)$ are quasi-analytic. Then we recover $\varphi(t,z)$ from the initial condition $v(x,0)=h(x)$ and we recover $v(x,t)$ from $\varphi(t,z)$.

Similar results are valid when the quasi-analytic initial condition is recovered from boundary conditions. (See ALS, J. Math. Anal. Appl. (2015).)

Next, we discuss interconnections between Dirac (spectral Dirac) and dynamical Dirac system. It suffices for our purposes to consider the case of square Weyl functions from Herglotz class (and locally bounded potentials V). The more general case (which we discussed earlier) is treated in the same way.

Recall that Weyl function of the spectral Dirac system

$$y_x = \mathrm{i}\big(zj + jV(x)\big)y(x,z), \quad j = \begin{bmatrix} I_k & 0 \\ 0 & -I_k \end{bmatrix} \tag{19}$$

is uniquely determined by the inequality

$$\int_0^\infty \begin{bmatrix} I_k & \mathrm{i}\varphi_H(z)^* \end{bmatrix} Y(x,z)^* Y(x,z) \begin{bmatrix} I_k \\ -\mathrm{i}\varphi_H(z) \end{bmatrix} dx < \infty$$

for all $z \in \mathbb{C}_+$, where Y is the fundamental solution of (19) normalized by

$$Y(0,z) = \Theta^*, \quad \Theta := \frac{1}{\sqrt{2}} \begin{bmatrix} I_k & -I_k \\ I_k & I_k \end{bmatrix}.$$

On the next frame we describe the recovery of V from φ_H.

Introduce convolution operators $S_\ell > 0$ acting in $L_k^2(0, \ell)$:

$$S_\ell = \frac{d}{dx} \int_0^\ell s(x-t) \cdot dt, \qquad s(x) = -s(-x)^*, \tag{20}$$

$$s(x)^* := \frac{d}{dx} \left(\frac{\mathrm{i}}{4\pi} \mathrm{e}^{\eta x} \mathrm{l.i.m.}_{a\to\infty} \int_{-a}^{a} \mathrm{e}^{-\mathrm{i}\xi x} (\xi + \mathrm{i}\eta)^{-2} \varphi_H(\xi + \mathrm{i}\eta) d\xi \right),$$

where l.i.m is the entrywise limit in $L^2(0, \infty)$ norm. Then

$$V(x) = \begin{bmatrix} 0 & v(x) \\ v(x)^* & 0 \end{bmatrix}, \quad v(x) = \mathrm{i}\theta_1'(x) J \theta_2(x)^*, \quad \text{where}$$

$$\theta_2(x) = \frac{1}{\sqrt{2}} \left(\begin{bmatrix} -I_k & I_k \end{bmatrix} - \int_0^{2x} \omega(t)^* S_{2x}^{-1} \begin{bmatrix} 2s(t) & I_k \end{bmatrix} dt \right),$$

$\omega(t) = s'(t)$ and θ_1 is uniquely determined by θ_2 and equalities

$$\theta_1(0) = \frac{1}{\sqrt{2}} \begin{bmatrix} I_k & I_k \end{bmatrix}, \quad \theta_1(x) J \theta_2(x)^* \equiv 0, \quad \theta_1'(x) J \theta_1(x)^* \equiv 0.$$

Here s' is an accelerant in the terminology of M.G. Krein and $2\mathrm{i}(s')^*$ is an analogue of the A-amplitude in Gesztesy-Simon terminology.

Dynamical Dirac system has the following form:

$$\mathrm{i}u_t + Ju_x + \mathcal{V}u = 0 \quad (x > 0, \quad t > 0); \tag{21}$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathcal{V}(x) = \begin{bmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{bmatrix},$$

where p and q are real-valued functions and initial-boundary conditions are given by the equalities

$$u(x,0) = 0, \; x \geq 0; \quad u_1(0,t) = f(t), \; t \geq 0. \tag{22}$$

Here f is a complex-valued function (boundary control) and the input-output map $R: \; u_1(0,\cdot) \to u_2(0,\cdot)$ is of the convolution form $Rf = \mathrm{i}f + r * f$. The inverse problem consists in recovery of the potential $\mathcal{V}$ from the *response function* r.

For continuously differentiable $\mathcal{V}$, f, r, this problem is solved in M. Belishev, V. Mikhailov, Inverse Problems 30 (2014), 125013 using boundary control methods.

Under natural conditions, we can take Fourier transformation of u and r and come to the spectral Dirac system and Weyl function, respectively.

Namely, we have

$$\widehat{u}(x,z) := \int_0^\infty \mathrm{e}^{\mathrm{i}zt} u(x,t)dt, \quad \widehat{r}(z) = \int_0^\infty \mathrm{e}^{\mathrm{i}zt} r(t)dt \qquad (23)$$

$$(z \in \mathbb{C}_M = \{z : \Im(z) > M\}).$$

Then $\widehat{u}$ satisfies spectral Dirac system in an equivalent (to the one given before in the talk) form

$$z\widehat{u}(x,z) + J\widehat{u}_x(x,z) + \mathcal{V}(x)\widehat{u}(x,z) = 0 \qquad (24)$$

and there is a simple correspondence between $\widehat{r}$ and the Weyl function φ_H:

$$\varphi_H(z) = \widehat{r}(z) + \mathrm{i}. \qquad (25)$$

Initial-boundary conditions don't need reduction for the classical sine-Gordon eq-n

$$\psi_{xt} = 2\sin(2\psi). \tag{26}$$

Local solutions of the in.-bound. probl. for (26) were studied in I.M. Krichever, Dokl. Akad. Nauk. SSSR, **253**:2, 288–292 (1980); A.N. Leznov, M.V. Saveliev, Progress in Physics 15, Birkhäuser.

For global solutions see ALS, Russ. Math. Iz. VUZ **36** (1992) and ALS, Nonlinear Analysis **75**, 964–974 (2012).

Now, system $W_x = GW$ is a skew-selfadjoint Dirac system, we use gw-functions (generalized Weyl functions) instead of Weyl functions and evolution of the gw-function is again described via Möbius transformation.

Theorem 5. Let the initial–boundary conditions

$$\psi(x,0) = h_1(x), \quad \psi(0,t) = h_2(t), \quad h_1(0) = h_2(0), \tag{27}$$

$(h_k = \overline{h_k})$ be given. Assume that h_2 is continuous on $[0,\, a)$ and that h_1 is boundedly differentiable on all the finite intervals on $[0,\, \infty)$. Assume also that the gw-function $\varphi(0,z)$ of the system

$$\frac{d}{dx}W = GW, \quad G(x,z) = \mathrm{i}zj - \begin{bmatrix} 0 & h_1'(x) \\ -h_1'(x) & 0 \end{bmatrix} \tag{28}$$

exists and satisfies

$$\sup |z^2(\varphi(z) - \phi_0/z)| < \infty \quad (\Im z > M > 0), \tag{29}$$

where ϕ_0 is some constant. Then a solution of the initial-boundary value problem (27) for $\psi_{xt} = 2\sin(2\psi)$ exists and is given by

$$\psi(x,t) = h_2(t) - \int_0^x \Big(\mathfrak{M}\big(\varphi(t,z)\big)\Big)(\xi)d\xi, \tag{30}$$

where $\varphi(t,z)$ is constructed in Theorem 3 and $\mathfrak{M}\big(\varphi(t,z)\big)$ is the solution v of the inverse problem, which is recovered from φ.