On the pseudo-Frobenius numbers of numerical semigroups

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* $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
* $\mathbb{N} = \{0, 1, 2, \dots\}$

Definition

- *S numerical semigroup = S cofinite submonoid of* $(\mathbb{N}, +)$.

Definition

- $x \notin S$ *is the* Frobenius number *of* S *if* $x + s \in S$ *for all* $s \in \mathbb{N} \setminus \{0\}$.
- $x \notin S$ *is a* pseudo-Frobenius number *of* S *if* $x + s \in S$ *for all* $s \in S \setminus \{0\}$.

* $\mathrm{PF}(S) = \{$Pseudo-Frobenius numbers of $S\}$
* $\mathrm{t}(S) = \sharp\mathrm{PF}(S)$ = type of S
* $\mathrm{F}(S) =$ Frobenius number of S
* $\mathrm{F}(S) = \max(\mathrm{PF}(S)) = \max(\mathbb{Z} \setminus S)$
* $\mathrm{m}(S) = \min(S \setminus \{0\})$

Motivation (1)

- $f \in \mathbb{N} \setminus \{0\}$
- $\mathcal{S}(f) = \{S \text{ numerical semigroup } \mid \mathrm{F}(S) = f\}$
- $\mathcal{S}(f) \neq \emptyset$ for all $f \in \mathbb{N} \setminus \{0\}$
- There are algorithms to compute $\mathcal{S}(f)$

  (Rosales, García-Sánchez, García-García, Jiménez-Madrid, J. Pure Appl. Algebra **189**, 2004)

** $\mathrm{PF} = \{g_1, g_2, \ldots, g_{n-1}, g_n\} \subseteq \mathbb{N} \setminus \{0\}$

** $\mathcal{S}(\mathrm{PF}) = \{S \text{ numerical semigroup } \mid \mathrm{PF}(S) = \mathrm{PF}\}$

** $\mathcal{S}(\mathrm{PF}) \neq \emptyset$ if $n = 1$

** $\mathcal{S}(\mathrm{PF})$ may be empty if $n \geq 2$

** $\mathcal{S}(\mathrm{PF})$ is finite when it is non-empty

Motivation (2)

Questions

1. Find conditions on the set PF that ensure that $\mathcal{S}(\mathrm{PF}) \neq \emptyset$.
2. Find an algorithm to compute $\mathcal{S}(\mathrm{PF})$.

Solved cases

- $\mathrm{PF} = \{f\}$: symmetric numerical semigroups.
  (Blanco, Rosales, Forum Math. **25**, 2013)
- $\mathrm{PF} = \{f, \frac{f}{2}\}$: pseudo-symmetric numerical semigroups.
  (Blanco, Rosales, Forum Math. **25**, 2013)
- Question 1 for $\mathrm{PF} = \{g_1, g_2\}$.
  (Robles-Pérez, Rosales, preprint)

Motivation (3)

Proposition

- *Let g_1, g_2 be two positive integers such that $g_1 < g_2$. Let us take*
- *Let* $g^* = \begin{cases} 2g_1 - g_2 & \text{if } g_2 \text{ is odd,} \\ g_1 - \frac{g_2}{2} & \text{if } g_2 \text{ is even.} \end{cases}$
- *There exists a numerical semigroup S such that $\mathrm{PF}(S) = \{g_1, g_2\}$ if and only if*
$$\frac{g_2}{2} < g_1,\ g^* \nmid g_1,\ g^* \nmid g_2,\ g^* \nmid (g_2 - g_1).$$

Example

- $\mathrm{PF} = \{11, 15\}$ is an admissible set of pseudo-Frobenius numbers.
- $S = \langle 6, 7, 10 \rangle$ and $S = \langle 7, 9, 10, 12, 13 \rangle$.

Motivation (4)

Example

- $\mathrm{PF} = \{9, 15\}$ is not a set of pseudo-Frobenius numbers.
  $(g^* = 2 \times 9 - 15 = \mathbf{3})$
- $\mathrm{PF} = \{9, 10, 15\}$ is a set of pseudo-Frobenius numbers.
  $S = \langle 4, 13, 14, 19 \rangle$.
  $(g_1^* = 2 \times 9 - 15 = \mathbf{3},\ g_2^* = 2 \times 9 - 10 = 8,\ g_3^* = 2 \times 10 - 15 = \mathbf{5})$

Example

- $\mathrm{PF} = \{12, 18\}$ is not a set of pseudo-Frobenius numbers.
  $(g^* = 12 - \frac{18}{2} = \mathbf{3})$
- $\mathrm{PF} = \{12, 16, 18\}$ is a set of pseudo-Frobenius numbers.
  $S = \langle 7, 10, 13, 15, 19 \rangle$.
  $(g_1^* = 12 - \frac{18}{2} = \mathbf{3},\ g_2^* = 12 - \frac{16}{2} = \mathbf{4},\ g_3^* = 16 - \frac{18}{2} = 7)$

A simple idea to compute $\mathcal{S}$(PF): by filtering $\mathcal{S}$(max(PF))

```
gap> pf := [17,27];;
gap> nsf27 := NumericalSemigroupsWithFrobeniusNumber(27);;time;
12062
gap> Length(nsf27);
16132
gap> nspf1727 := Filtered(nsf27, s -> PseudoFrobeniusOfNumerical
Semigroup(s) = pf);;time;Length(nspf1727);
1313
5

gap> pf := [19,29];;
gap> nsf29 := NumericalSemigroupsWithFrobeniusNumber(29);;time;
43281
gap> Length(nsf29);
34903
gap> nspf1929 := Filtered(nsf29, s -> PseudoFrobeniusOfNumerical
Semigroup(s) = pf);;time;Length(nspf1929);
3171
13
```

Alternatives

$*$ Determine the elements and the gaps (“forced” integers) of all numerical semigroups with the given set of pseudo-Frobenius numbers, obtaining a list of “free” integers. We then construct a tree in which branches correspond to assuming that free integers are either gaps or elements.

$**$ Use irreducible numerical semigroups: from each irreducible numerical semigroup we start removing minimal generators with certain properties to build a tree whose leafs are the numerical semigroups we are looking for.

Forced and free integers

- $\mathrm{PF} = \{g_1, g_2, \ldots, g_{n-1}, g_n\}$ a given set ($\subseteq \mathbb{N} \setminus \{0\}$)
- $\ell \in \mathcal{GF}(\mathrm{PF})$ if ℓ is a gap of any $S \in \mathcal{S}(\mathrm{PF})$
- $\ell \in \mathcal{EF}(\mathrm{PF})$ if $\ell \leq g_n + 1$ and it is an element of any $S \in \mathcal{S}(\mathrm{PF})$

Proposition

- $\mathcal{S}(\mathrm{PF}) \neq \emptyset$ *if and only if* $\mathcal{GF}(\mathrm{PF}) \cap \mathcal{EF}(\mathrm{PF}) = \emptyset$.

** $\mathit{Free}(\mathrm{PF}) = \{1, \ldots, g_n\} \setminus (\mathcal{GF}(\mathrm{PF}) \cup \mathcal{EF}(\mathrm{PF}))$

Example

- $\mathrm{PF} = \{4, 17, 21\}$
- $\mathcal{GF}(\mathrm{PF}) = \{1, 2, 3, 4, 5, 7, 9, 13, 17, 21\}$
- $\mathcal{EF}(\mathrm{PF}) = \{8, 12, 14, 16, 18, 19, 20\}$
- $\mathit{Free}(\mathrm{PF}) = \{6, 10, 11, 15\}$

Starting forced gaps (1)

Lemma

- *Let S be a numerical semigroup.*
- *Then $\mathrm{m}(S) \geq \mathrm{t}(S) + 1$.*

Lemma

- *Let $\mathrm{PF}(S) = \{g_1 < g_2 < \cdots < g_{n-1} < g_n\}$, with $n > 1$.*
- *Let $i \in \{2, \ldots, \mathrm{t}(S)\}$ and $g \in \langle \mathrm{PF}(S) \rangle$ with $g < g_i$.*
- *Then $g_i - g \in gaps(S) = \mathbb{N} \setminus S$.*

* The set of positive divisors of

$$\mathrm{PF}(S) \cup \{x \in \mathbb{N} \mid 1 \leq x \leq \mathrm{t}(S)\} \cup$$

$$\{g_i - g \mid i \in \{2, \ldots, \mathrm{t}(S)\}, g \in \langle \mathrm{PF}(S) \rangle, g_i > g\}$$

is a subset of $gaps(S)$.

Starting forced gaps (2)

- $\mathrm{PF} = \{g_1 < g_2 < \cdots < g_{n-1} < g_n\}$ $(n > 1)$
- Let $\mathrm{sfg}(\mathrm{PF})$ be the set of positive divisors of

$$\mathrm{PF} \cup \{x \in \mathbb{N} \mid 1 \leq x \leq n\} \cup \{g_i - g \mid i \in \{2, \ldots, n\}, g \in \langle \mathrm{PF} \rangle, g_i > g\}.$$

Corollary

- *Let* PF *be a set of positive integers.*
- *Then* $\mathrm{sfg}(\mathrm{PF}) \subseteq \mathcal{GF}(\mathrm{PF})$.

Example

- $\mathrm{PF} = \{4, 9\} \Rightarrow \mathrm{sfg}(\mathrm{PF}) = \{1, 2, 3, 4, 5, 9\}$
- $\mathrm{PF} = \{11, 17\} \Rightarrow \mathrm{sfg}(\mathrm{PF}) = \{1, 2, 3, 6, 11, 17\}$

Initial necessary condition

Lemma

- *Let S be a numerical semigroup.*
- *Then $x \in \mathbb{Z} \setminus S$ if and only if $f - x \in S$ for some $f \in \mathrm{PF}(S)$.*

Lemma

- *Let $\mathrm{PF}(S) = \mathsf{PF} = \{g_1 < \cdots < g_n\}$.*
- *Let $i \in \{2, \ldots, n\}$ and $g \in \langle \mathsf{PF} \rangle \setminus \{0\}$ with $g < g_i$.*
- *Then there exists $k \in \{1, \ldots, n\}$ such that $g_k - (g_i - g) \in S$.*

Corollary

- $g_1 \geq g_n - g_{n-1}$.

Example

- $\mathsf{PF} = \{4, 9\} \Rightarrow \mathcal{S}(\mathsf{PF}) = \emptyset$

Forced elements: big forced elements

Lemma

- *Let S be a numerical semigroup such that $\mathrm{m}(S) = m$.*
- *Then $\mathrm{F}(S) - i \in S \cup \mathrm{PF}(S)$ for all $i \in \{1, \ldots, m\}$.*

Consequence

- Let $\mathrm{fg}(\mathrm{PF}) \subseteq \mathcal{GF}(\mathrm{PF})$.
- Let $m = \min(\mathbb{N} \setminus (\mathrm{fg}(\mathrm{PF}) \cup \{0\}))$.
- Then $\max(\mathrm{PF}) - i \in \mathcal{EF}(\mathrm{PF}) \cup \mathrm{PF}$ for all $i \in \{1, \ldots, m\}$.

Example

- $\mathrm{PF} = \{4, 9\} \Rightarrow \mathrm{fg}(\mathrm{PF}) = \mathrm{sfg}(\mathrm{PF}) = \{1, 2, 3, 4, 5, 9\} \Rightarrow \mathrm{bfe}(\mathrm{PF}) = \{5, 6, 7, 8\}$ Contradiction!
- $\mathrm{PF} = \{11, 17\} \Rightarrow \mathrm{fg}(\mathrm{PF}) = \mathrm{sfg}(\mathrm{PF}) = \{1, 2, 3, 6, 11, 17\} \Rightarrow \mathrm{bfe}(\mathrm{PF}) = \{14, 15, 16\}$

Forced elements: elements forced by exclusion (1)

Lemma

- *Let S be a numerical semigroup.*
- *Then $x \in \mathbb{Z} \setminus S$ if and only if $f - x \in S$ for some $f \in \mathrm{PF}(S)$.*

Consequence

- Let $x \in \mathrm{fg}(\mathsf{PF}) \subseteq \mathcal{GF}(\mathsf{PF})$.
- If there exists a unique $f^* \in \mathrm{PF}(S)$ such that $f^* - x \notin \mathrm{fg}(\mathsf{PF})$, ...
- ... then $f^* - x \in \mathrm{efe}_1(\mathsf{PF}) \subseteq \mathcal{EF}(\mathsf{PF})$.

Example

- $\mathsf{PF} = \{11, 17\} \Rightarrow \mathrm{fg}(\mathsf{PF}) = \mathrm{sfg}(\mathsf{PF}) = \{1, 2, 3, 6, 11, 17\} \Rightarrow$ $\mathrm{efe}_1(\mathsf{PF}) = \{5\}$

Forced elements: elements forced by exclusion (2)

Lemma

- *Let S be a numerical semigroup.*
- *Then $x \in S$ if and only if $f - x \notin S$ for all $f \in \mathrm{PF}(S)$.*

Consequence

- Let $x \notin \mathrm{fg}(\mathsf{PF}) \cup \mathrm{bfe}(\mathsf{PF}) \cup \mathrm{efe}_1(\mathsf{PF})$.
- If $f - x \in \mathrm{fg}(\mathsf{PF})$, for every $f \in \mathsf{PF}$, ...
- then $x \in \mathrm{efe}_2(\mathsf{PF}) \subseteq \mathcal{EF}(\mathsf{PF})$.

Example

- $\mathsf{PF} = \{11,17\} \Rightarrow \mathrm{fg}(\mathsf{PF}) = \mathrm{sfg}(\mathsf{PF}) = \{1,2,3,6,11,17\} \Rightarrow$ $\mathrm{bfe}(\mathsf{PF}) \cup \mathrm{efe}_1(\mathsf{PF}) = \{5,14,15,16\} \Rightarrow \mathrm{efe}_2(\mathsf{PF}) = \emptyset$

$*$ $\mathrm{fe}(\mathsf{PF}) = \langle \mathrm{bfe}(\mathsf{PF}) \cup \mathrm{efe}_1(\mathsf{PF}) \cup \mathrm{efe}_2(\mathsf{PF}) \rangle \cap [1, \max(\mathsf{PF})] \subseteq \mathcal{EF}(\mathsf{PF})$

Further forced gaps

Remark

- *Let S be a numerical semigroup.*
- *Let $e \in S$ and $f \in gaps(S)$.*
- *Then $f - e < 0$ or $f - e \in gaps(S)$.*

Consequence

- Let $\mathrm{fg}(\mathrm{PF}) \subseteq \mathcal{GF}(\mathrm{PF})$ and $\mathrm{fe}(\mathrm{PF}) \subseteq \mathcal{EF}(\mathrm{PF})$.
- Then $\Delta = (\mathrm{fg}(\mathrm{PF}) - e) \cap \mathbb{N} \subseteq \mathrm{ffg}(\mathrm{PF}) \subseteq \mathcal{GF}(\mathrm{PF})$ for every $e \in \mathrm{fe}(\mathrm{PF})$.

Example

- $\mathrm{PF} = \{11, 17\} \Rightarrow \mathrm{sfg}(\mathrm{PF}) = \{1, 2, 3, 6, 11, 17\} \Rightarrow$ $\mathrm{fe}(\mathrm{PF}) = \{5, 10, 14, 15, 16\} \Rightarrow \Delta = \{7, 12\}$

$*$ $\mathrm{ffg}(\mathrm{PF}) = \{\text{Positive divisors of } \mathrm{sfg}(\mathrm{PF}) \cup \Delta\} \subseteq \mathcal{GF}(\mathrm{PF})$

Algorithm 1

* PF

* PF $\Rightarrow \mathrm{sfg}(\mathrm{PF})$
* PF, $\mathrm{sfg}(\mathrm{PF}) \Rightarrow \mathrm{fe}(\mathrm{PF})$

* PF, $\mathrm{sfg}(\mathrm{PF}), \mathrm{fe}(\mathrm{PF}) \Rightarrow \mathrm{ffg}_1(\mathrm{PF})$
* PF, $\mathrm{ffg}_1(\mathrm{PF}), \mathrm{fe}(\mathrm{PF}) \Rightarrow \mathrm{ffe}_1(\mathrm{PF})$

* PF, $\mathrm{ffg}_1(\mathrm{PF}), \mathrm{ffe}_1(\mathrm{PF}) \Rightarrow \mathrm{ffg}_2(\mathrm{PF})$
* PF, $\mathrm{ffg}_2(\mathrm{PF}), \mathrm{ffe}_1(\mathrm{PF}) \Rightarrow \mathrm{ffe}_2(\mathrm{PF})$

$\vdots$

* Repeat until $\mathrm{ffg}_{k+1}(\mathrm{PF}) = \mathrm{ffg}_k(\mathrm{PF})$ and $\mathrm{ffe}_{k+1}(\mathrm{PF}) = \mathrm{ffe}_k(\mathrm{PF})$

** $\mathrm{ffg}_k(\mathrm{PF}) \subseteq \mathcal{GF}(\mathrm{PF}),\ \mathrm{ffe}_k(\mathrm{PF}) \subseteq \mathcal{EF}(\mathrm{PF})$

** $[1, \max(\mathrm{PF})] \setminus (\mathrm{ffg}_k(\mathrm{PF}) \cup \mathrm{ffe}_k(\mathrm{PF})) \supseteq \mathit{Free}(\mathrm{PF})$

Example

- $\ast$ $\mathsf{PF} = \{11, 17\}$
- $\ast$ $\mathsf{PF} = \{11, 17\} \Rightarrow \mathrm{sfg}(\mathsf{PF}) = \{1, 2, 3, 6, 11, 17\}$
- $\ast$ $\mathsf{PF} = \{11, 17\}, \mathrm{sfg}(\mathsf{PF}) = \{1, 2, 3, 6, 11, 17\} \Rightarrow \mathrm{fe}(\mathsf{PF}) = \{5, 10, 14, 15, 16\}$
- $\ast$ $\mathsf{PF} = \{11, 17\}, \mathrm{sfg}(\mathsf{PF}) = \{1, 2, 3, 6, 11, 17\} \mathrm{fe}(\mathsf{PF}) = \{5, 10, 14, 15, 16\} \Rightarrow \mathrm{ffg}_1(\mathsf{PF}) = \{1, 2, 3, 4, 6, 7, 11, 12, 17\}$
- $\ast$ $\mathsf{PF} = \{11, 17\}, \mathrm{ffg}_1(\mathsf{PF}) = \{1, 2, 3, 4, 6, 7, 11, 12, 17\}, \mathrm{fe}(\mathsf{PF}) = \{5, 10, 14, 15, 16\} \Rightarrow \mathrm{ffe}_1(\mathsf{PF}) = \{5, 10, 13, 14, 15, 16\}$
- $\ast$ $\mathsf{PF} = \{11, 17\}, \mathrm{ffg}_1(\mathsf{PF}) = \{1, 2, 3, 4, 6, 7, 11, 12, 17\}, \mathrm{ffe}_1(\mathsf{PF}) = \{5, 10, 13, 14, 15, 16\} \Rightarrow \mathrm{ffg}_2(\mathsf{PF}) = \{1, 2, 3, 4, 6, 7, 11, 12, 17\}$
- $\ast$ $\mathsf{PF} = \{11, 17\}, \mathrm{ffg}_2(\mathsf{PF}) = \{1, 2, 3, 4, 6, 7, 11, 12, 17\}, \mathrm{ffe}_1(\mathsf{PF}) = \{5, 10, 13, 14, 15, 16\} \Rightarrow \mathrm{ffe}_2(\mathsf{PF}) = \{5, 10, 13, 14, 15, 16\}$

- $\ast\ast$ $\mathrm{ffg}_2(\mathsf{PF}) = \mathcal{GF}(\mathsf{PF}) = \{1, 2, 3, 4, 6, 7, 11, 12, 17\}$
- $\ast\ast$ $\mathrm{ffe}_2(\mathsf{PF}) = \mathcal{EF}(\mathsf{PF}) = \{5, 10, 13, 14, 15, 16\}$
- $\ast\ast$ $\mathit{Free}(\mathsf{PF}) = \{8, 9\}$

Example

$*$ $\text{PF} = \{11, 17\}$, $\text{ffg}_2(\text{PF}) = \mathcal{GF}(\text{PF}) = \{1, 2, 3, 4, 6, 7, 11, 12, 17\}$
$\text{ffe}_2(\text{PF}) = \mathcal{EF}(\text{PF}) = \{5, 10, 13, 14, 15, 16\}$, $\mathit{Free}(\text{PF}) = \{8, 9\}$

$**$ Algorithm 1 with $\text{ffg}(\text{PF}) = \{1, 2, 3, 4, 6, 7, 11, 12, 17\}$ and
$\text{ffe}(\text{PF}) = \{5, 8, 10, 13, 14, 15, 16\}$
$S = \langle 5, 8, 14 \rangle = \{0, 5, 8, 10, 13, 14, 15, 16, 18, \rightarrow\}$

$**$ Algorithm 1 with $\text{ffg}(\text{PF}) = \{1, 2, 3, 4, 6, 7, 8, 11, 12, 17\}$ and
$\text{ffe}(\text{PF}) = \{5, 10, 13, 14, 15, 16\}$
$S = \langle 5, 9, 13, 16 \rangle = \{0, 5, 9, 10, 13, 14, 15, 16, 18, \rightarrow\}$

Example

- $\mathsf{PF} = \{15, 20, 27, 35\}$

- $\mathrm{sfg}(\mathsf{PF}) = \{1,2,3,4,5,6,7,8,9,10,12,15,20,27,35\}$
- $\mathrm{fe}(\mathsf{PF}) = \{19,23,25,26,28,29,30,31,32,33,34\}$

- $\mathrm{ffg}_1(\mathsf{PF}) = \{1,2,3,4,5,6,7,8,9,10,12,15,16,20,27,35\}$
- $\mathrm{ffe}_1(\mathsf{PF}) = \{19,23,25,26,28,29,30,31,32,33,34\}$

- $\mathrm{ffg}_2(\mathsf{PF}) = \{1,2,3,4,5,6,7,8,9,10,12,15,16,20,27,35\}$
- $\mathrm{ffe}_2(\mathsf{PF}) = \{19,23,25,26,28,29,30,31,32,33,34\}$

** $\mathrm{ffg}_2(\mathsf{PF}) = \{1,2,3,4,5,6,7,8,9,10,12,15,20,27,35\} \subseteq \mathcal{GF}(\mathsf{PF})$

** $\mathrm{ffe}_2(\mathsf{PF}) = \{19,23,25,26,28,29,30,31,32,33,34\} \subseteq \mathcal{EF}(\mathsf{PF})$

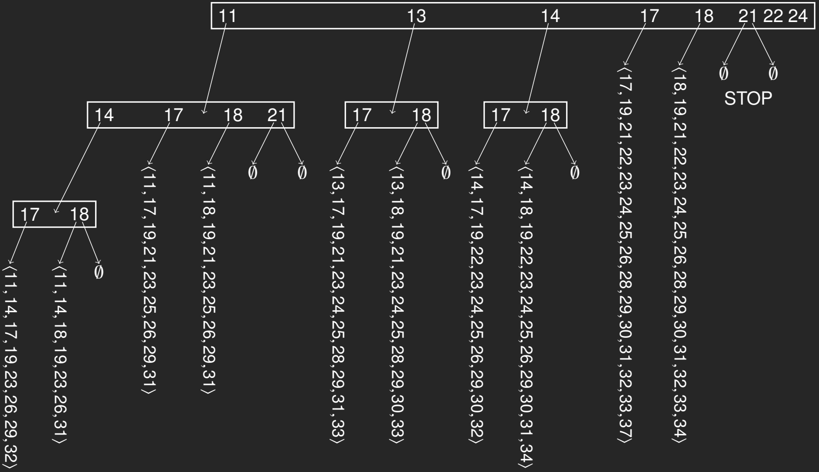
** $\mathit{Free}(\mathsf{PF}) \subseteq \{8,9\}$

Example: PF $= \{15, 20, 27, 35\}$

$\{1,2,3,4,5,6,7,8,9,10,12,15,16,20,27,35\} \subseteq \mathcal{GF}(\text{PF})$

$\{19,23,25,26,28,29,30,31,32,33,34\} \subseteq \mathcal{EF}(\text{PF})$

$\mathit{Free}(\text{PF}) \subseteq \{11,13,14,17,18,21,22,24\}$



Numerical semigroups with pseudo-Frobenius numbers $\{15, 20, 27, 35\}$

Running times

```
gap> pf:=[11,17];;
gap> ans:=NumericalSemigroupsWithPseudoFrobeniusNumbers(pf);;
gap> time;Length(ans);
0
2

gap> pf:=[17,27];;
gap> ans:=NumericalSemigroupsWithPseudoFrobeniusNumbers(pf);;
gap> time;Length(ans);
0
5

gap> pf:=[19,29];;
gap> ans:=NumericalSemigroupsWithPseudoFrobeniusNumbers(pf);;
gap> time;Length(ans);
16
13
```

Using irreducible numerical semigroups

Proposition

Let S be a nonirreducible numerical semigroup with $\mathrm{PF}(S) = \{g_1 < \cdots < g_k\}$, $k \geq 2$. Then there exists a chain

$$S = S_0 \subset S_1 = S \cup \{x_1\} \subset \cdots \subset S_l = S \cup \{x_1, \ldots, x_l\},$$

with

(1) *S_l irreducible,*

(2) *$x_i = \max(S_l \setminus S_{i-1})$ for all $i \in \{1, \ldots, l\}$,*

(3) *$(g_k/2, g_k) \cap \mathrm{PF}(S) \subset S_l$,*

(4) *for every $i \in \{1, \ldots, l\}$, x_i is a minimal generator of S_i such that $g_j - x_i \in S_i$ for some $j \in \{1, \ldots, k\}$,*

(5) *for every $i \in \{1, \ldots, l\}$ and $f \in \mathrm{PF}(S_i)$ with $f \neq g_k$, there exists $j \in \{1, \ldots, k-1\}$ such that $g_j - f \in S_i$.*

Example

- Irreducible numerical semigroups with $\mathrm{F}(T_i) = 17$
  - $T_1 = \langle 2, 19 \rangle$
  - $T_2 = \langle 3, 10 \rangle$
  - $T_3 = \langle 4, 6, 15 \rangle$
  - $T_4 = \langle 4, 7 \rangle$
  - $T_5 = \langle 4, 10, 11 \rangle$
  - $T_6 = \langle 5, 8, 11, 14 \rangle$
  - $T_7 = \langle 5, 9, 11, 13 \rangle$
  - $T_8 = \langle 6, 7, 8 \rangle$
  - $T_9 = \langle 6, 7, 9 \rangle$
  - $T_{10} = \langle 6, 8, 10, 13, 15 \rangle$
  - $T_{11} = \langle 6, 9, 10, 13, 14 \rangle$
  - $T_{12} = \langle 7, 8, 11, 12, 13 \rangle$
  - $T_{13} = \langle 7, 9, 11, 12, 13, 15 \rangle$
  - $T_{14} = \langle 8, 10, 11, 12, 13, 14, 15 \rangle$
  - $T_{15} = \langle 9, 10, 11, 12, 13, 14, 15, 16 \rangle$
- Numerical semigroups with $\mathrm{PF}(S_i) = \{11, 17\}$
  - $S_1 = T_6 \setminus \{11\} = \langle 5, 8, 14 \rangle$
  - $S_2 = T_7 \setminus \{11\} = \langle 5, 9, 13, 16 \rangle$

THANK YOU VERY MUCH FOR YOUR ATTENTION!