Regularity of Vanishing Ideals associated to Graphs

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Outline

Parameterized Algebraic Toric Sets

Let G be a simple graph with n vertices $\{y_1, \ldots, y_n\}$ and s edges. Let

$$y_{1_1}y_{1_2}\ ,\ \ldots\ ,\ y_{s_1}y_{s_2}$$

be the set of all monomials $y_k y_l$ such that $\{y_k, y_l\}$ is an edge of G.

Let $K = \mathbb{F}_q$ be a finite field with q elements.

$$X := \{\ [\ (\ x_{1_1}x_{1_2}\ ,\ \ldots\ ,\ x_{s_1}x_{s_2}\ )\ ]\ \in \mathbb{P}^{s-1}\ :\ x_{i_j} \in K \setminus \{0\}\ \}$$

is called the *algebraic toric set parameterized* by the edges of G.

$$X = \{[P_1], \ldots, [P_m]\} \subseteq \mathbb{P}^{s-1}\ .$$

Parameterized Linear Codes

Let $S = K[t_1, \ldots, t_s]$ be a polynomial ring with the standard grading: $S = \bigoplus_{d=0}^{\infty} S_d$.

For each $d \geq 0$, we have a linear map of K-vector spaces:

$$\mathrm{ev}_d \colon K[t_1, \ldots, t_s]_d \to K^m$$

$$f \mapsto \left(\frac{f(P_1)}{f_0(P_1)}, \ldots, \frac{f(P_m)}{f_0(P_m)} \right), \qquad \text{where } f_0(t_1 \ldots, t_s) = t_1^d.$$

The image of ev_d, denoted by C_d, is called the *parameterized linear code* of order d associated to the graph G.

Linear Codes

Definition

A *linear code* C is simply a linear subspace of K^m.

The basic parameters of a linear code are:

- m, the *length* of C
- $\dim_K C$, the *dimension* of C
- $\delta := \min\{\|v\| : 0 \neq v \in C\}$, the *minimum distance* of C, where $\|v\|$ is the number of non-zero entries of v.

The Singleton bound:

$$\delta \leq m - \dim_K C + 1.$$

The vanishing ideal $I(X)$

Definition

The vanishing ideal of X is the ideal $I(X) \subseteq S$, generated by the homogeneous polynomials vanishing on X.

The kernel of ev_d is precisely $I(X)_d$. Then there is an isomorphism of K-vector spaces

$$(S/I(X))_d \simeq C_d \ .$$

Theorem (Renteria, Simis, Villarreal, 2010):

- $I(X)$ is a radical, binomial ideal.
- $S/I(X)$ is Cohen-Macaulay and $\dim S/I(X) = 1$.

- $\dim_K C_d \;=\; \dim_K(S/I(X))_d \;=\; H(d)$,
  where H is the *Hilbert function* of $S/I(X)$.

There is an integer $r \geq 0$ such that
$1 = H(0) < H(1) < \cdots < H(r-1) < H(r) = |X| = m$
and for $d \geq r$, $\quad H(d) = H(r) = |X| = m$.

(Geramita, Kreuzer, Robbiano, 1993)

- length of $C_d \;= |X| = m$ is the *multiplicity* of $S/I(X)$.

Definition

Such integer $r \geq 0$ is called the *regularity index* of $S/I(X)$, denoted by $\operatorname{reg}(S/I(X))$, and in our case, it coincides with the Castelnuovo-Mumford regularity of $S/I(X)$.

- $d \geq \operatorname{reg}(S/I(X)) \;\Rightarrow\; \delta_d \leq m - H(d) + 1 = 1$.

The regularity $S/I(X)$

- If $X = \mathbb{T}^{s-1} = \{\ [(\alpha_1, \ldots, \alpha_s)] \in \mathbb{P}^{s-1} : \alpha_i \in K \setminus \{0\}\ \}$ is the *projective torus* of dimension $s-1$ over K, then $I(X) = (t_2^{q-1} - t_1^{q-1}, \ldots, t_s^{q-1} - t_1^{q-1})$ and

$$\operatorname{reg}(S/I(X)) = (s-1)(q-2)\ .$$

(González, Hernández, Rentería, 2003)

Complete intersections

- The following are equivalent:

i) $I(X)$ is a complete intersection
ii) $I(X) = (t_2^{q-1} - t_1^{q-1}, \ldots, t_s^{q-1} - t_1^{q-1})$
iii) $X = \mathbb{T}^{s-1}$

(Sarmiento, VP , Villarreal, 2010)

G is a non-bipartite graph

- If G is an odd cycle, X coincides with $\mathbb{T}^{s-1}$ and

$$\mathrm{reg}(S/I(X)) = (s-1)(q-2) = (n-1)(q-2)\,.$$

(Sarmiento, VP, Villarreal, 2011)

- In the case of a complete graph $G = \mathcal{K}_n$, if $n \geq 4$,

$$\mathrm{reg}(S/I(X)) = \lceil (n-1)(q-2)/2 \rceil\,.$$

(González, Rentería, Sarmiento, 2013)

- If $G = \mathcal{K}_{\alpha_1,\ldots,\alpha_r}$ is a complete multipartite graph with $n = \alpha_1 + \cdots + \alpha_r \geq 4$ vertices, $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r$

$$\mathrm{reg}(S/I(X)) = \max\{\,\alpha_1(q-2),\ \lceil (n-1)(q-2)/2 \rceil\,\}\,.$$

(Neves, VP, 2014)

G is a bipartite graph

- If $G = \mathcal{K}_{a_1,a_2}$ is a complete bipartite graph, $a_1 \geq a_2$, then
$$\operatorname{reg}(S/I(X)) = (a_1 - 1)(q - 2) .$$
(González, Rentería, 2008)

- If G is a tree, X coincides with $\mathbb{T}^{s-1}$ and
$$\operatorname{reg}(S/I(X)) = (s - 1)(q - 2) = (n - 2)(q - 2) .$$
(VP, Villarreal, 2012)

Theorem (N, VP, V):

- If $G = \mathcal{C}_{2k}$, an even cycle with $n = 2k$ vertices, then
$$\operatorname{reg}(S/I(X)) = (k - 1)(q - 2) .$$

The proof of the regularity

- t_1 is regular on $S/I(X)$

$$0 \longrightarrow (S/I(X))[-1] \xrightarrow{t_1} S/I(X) \longrightarrow S/(I(X), t_1) \longrightarrow 0$$

- $h_d := \dim_K(S/(I(X), t_1))_d = H(d) - H(d-1) \geq 0.$
- $r := (k-1)(q-2)$.

1. $h_d = 0$ for $d \geq r+1 \Rightarrow$ $H(d-1) = H(d)$ for $d-1 \geq r \Rightarrow \text{reg}(S/I(X)) \leq r$.
2. $h_r > 0 \Rightarrow H(r-1) < H(r) \Rightarrow \text{reg}(S/I(X)) \geq r.$

We need a set of generators for $I(X)$.

A generating set for $I(X)$

There is a set of generators for $I(X)$ consisting of

- the toric relations $t_i^{q-1} - t_j^{q-1}$,

  $i, j \in \{1, \ldots, s\},\ i \neq j$
  $(X \subseteq \mathbb{T}^{s-1} \Rightarrow I(\mathbb{T}^{s-1}) \subseteq I(X))$

  plus

- a finite set of homogeneous binomials $t^a - t^b$,

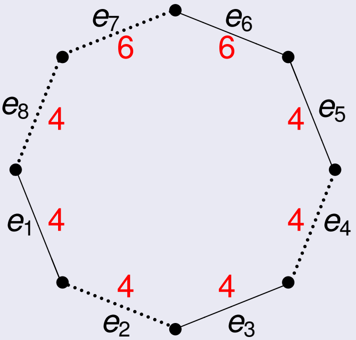
  $a = (a_1, \ldots, a_s),\ b = (b_1, \ldots, b_s) \in \mathbb{N}^s$, such that
  - $\mathrm{supp}(a) \cap \mathrm{supp}(b) = \emptyset$
  - $a_i,\ b_j \leq q - 2$
  - $\mathrm{supp}(a) \cup \mathrm{supp}(b) = \{1, \ldots, s\}$
  - $|\mathrm{supp}(a)| = |\mathrm{supp}(b)| = s/2$

A generating set for $I(X)$

Let $s = 8$ and choose $\sigma = A \cup B = \{1,3,5,6\} \cup \{2,4,7,8\}$.

Let $q = 11$. Choose $r = 4 \in \{1,\ldots,q-2\}$
and define $\hat{r} = (q-1) - r = 6$.

Define a function $\rho_\sigma^r : \{1,\ldots,s\} \to \{r, \hat{r}\}$, $\rho_\sigma^r(1) = r = 4$,



$\rho_\sigma^r(i+1) = \widehat{\rho_\sigma^r(i)}$,

if i and $i+1$ are on the same side of the partition,

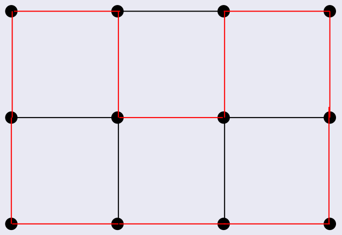
$\rho_\sigma^r(i+1) = \rho_\sigma^r(i)$, otherwise.

$$f_\sigma^r = t_1^4\, t_3^4\, t_5^4\, t_6^6 \; - \; t_2^4\, t_4^4\, t_7^6\, t_8^4$$

Definition

A cycle containing all the vertices of a graph is called a Hamilton cycle

$n = 12$ vertices, $k = 6$



Corollary

If G is a Hamiltonian bipartite graph with $2k$ vertices, then

$$\operatorname{reg}(S/I(X)) = (k-1)(q-2)$$

and the Eisenbud-Goto Regularity Conjecture holds for $I(X)$:

$$\operatorname{reg}(S/I(X)) \leq \deg(S/I(X)) - \operatorname{codim}(S/I(X)) .$$

What is the regularity of a block?

Definition

A vertex y of G is called a cutvertex if the number of connected components of $G \setminus \{y\}$ is larger than that of G.

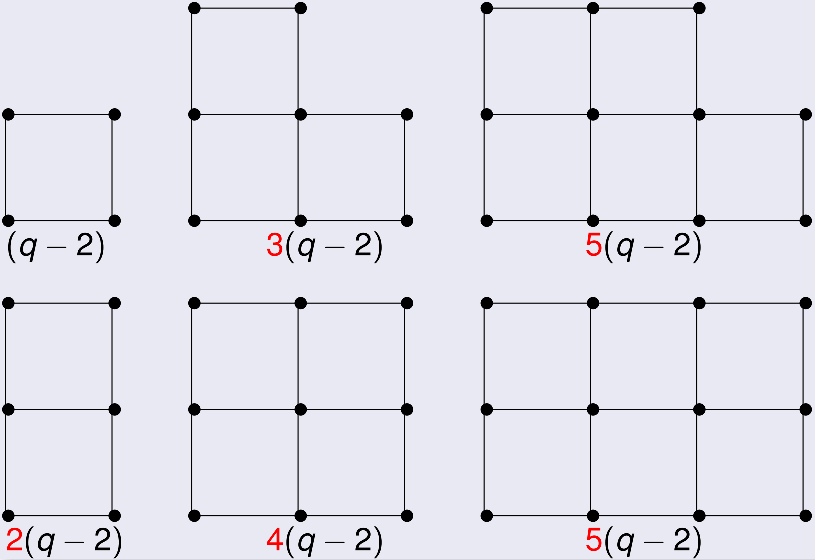
A maximal connected subgraph of G without cutvertices is called a block.

Theorem (N, VP, V)

Let G be a bipartite graph without isolated vertices and let $G_1, \ldots, G_c$ be the blocks of G. If X_k is the projective algebraic toric set parameterized by the edges of G_k, then

$$reg\, S/I(X) = \sum_{k=1}^{c} reg\, K[E_{G_k}]/I(X_k) + (c-1)(q-2).$$

$\mathrm{reg}(S/I(X))$



Conjecture

$G = \mathcal{C}_{2k_1} + \mathcal{C}_{2k_2}$, $k_1 \geq k_2$.

The two cycles are glued along a path with t edges.
Then

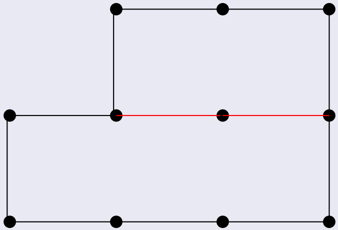
$$\operatorname{reg}(S/I(X)) = (k_1 + k_2 - 1 - \lceil t/2 \rceil)(q-2) ,$$

which is the regularity of the complete bipartite graph where G lives.

$G = \mathcal{C}_8 + \mathcal{C}_6 \subseteq \mathcal{K}_{6,5}$

$k_1 = 4,\ k_2 = 3$
$t = 2$

$\operatorname{reg} = 5(q-2)$



Thank you.