Quasi-split real groups and the Hitchin map

International Meeting AMS/EMS/SPM
Special session Higgs bundles and character varieties
Porto, June 2015

Ana Peón-Nieto

Mathematisches Institut
Ruprecht–Karls Universität
Heidelberg

Index

G-Higgs bundles

G-Higgs bundles

- X connected smooth projective curve/$\mathbb{C}$, $g(X) \geq 2$.

G-Higgs bundles

- X connected smooth projective curve/$\mathbb{C}$, $g(X) \geq 2$.
- G real reductive Lie group

G-Higgs bundles

- X connected smooth projective curve/$\mathbb{C}$, $g(X) \geq 2$.
- G real reductive Lie group $(\mathrm{GL}(n, \mathbb{R}),\ \mathrm{U}(n))$.

G-Higgs bundles

- X connected smooth projective curve/$\mathbb{C}$, $g(X) \geq 2$.
- G real reductive Lie group $(\mathrm{GL}(n, \mathbb{R}),\ \mathrm{U}(n))$.
- $H \leq G$ maximal compact subgroup

G-Higgs bundles

- X connected smooth projective curve/$\mathbb{C}$, $g(X) \geq 2$.
- G real reductive Lie group $(\mathrm{GL}(n, \mathbb{R}),\ \mathrm{U}(n))$.
- $H \leq G$ maximal compact subgroup $(\mathrm{O}(n),\ \mathrm{U}(n))$.

G-Higgs bundles

- X connected smooth projective curve/$\mathbb{C}$, $g(X) \geq 2$.
- G real reductive Lie group $(\mathrm{GL}(n, \mathbb{R}),\ \mathrm{U}(n))$.
- $H \leq G$ maximal compact subgroup $(\mathrm{O}(n),\ \mathrm{U}(n))$.
- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ Cartan decomposition,

G-Higgs bundles

- X connected smooth projective curve/$\mathbb{C}$, $g(X) \geq 2$.
- G real reductive Lie group $(\mathrm{GL}(n, \mathbb{R}),\ \mathrm{U}(n))$.
- $H \leq G$ maximal compact subgroup $(\mathrm{O}(n),\ \mathrm{U}(n))$.
- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ Cartan decomposition, θ Cartan involution.

G-Higgs bundles

- X connected smooth projective curve/$\mathbb{C}$, $g(X) \geq 2$.
- G real reductive Lie group $(\mathrm{GL}(n, \mathbb{R}),\ \mathrm{U}(n))$.
- $H \leq G$ maximal compact subgroup $(\mathrm{O}(n),\ \mathrm{U}(n))$.
- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ Cartan decomposition, θ Cartan involution.
- $H^{\mathbb{C}} \curvearrowright \mathfrak{m}^{\mathbb{C}}$ isotropy representation

G-Higgs bundles

- X connected smooth projective curve/$\mathbb{C}$, $g(X) \geq 2$.
- G real reductive Lie group $(\mathrm{GL}(n,\mathbb{R}),\ \mathrm{U}(n))$.
- $H \leq G$ maximal compact subgroup $(\mathrm{O}(n),\ \mathrm{U}(n))$.
- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ Cartan decomposition, θ Cartan involution.
- $H^{\mathbb{C}} \curvearrowright \mathfrak{m}^{\mathbb{C}}$ isotropy representation $(\mathrm{O}(n) \curvearrowright \mathfrak{sym}(n,\mathbb{R}))$.

G-Higgs bundles

- X connected smooth projective curve/$\mathbb{C}$, $g(X) \geq 2$.
- G real reductive Lie group $(\mathrm{GL}(n, \mathbb{R}),\ \mathrm{U}(n))$.
- $H \leq G$ maximal compact subgroup $(\mathrm{O}(n),\ \mathrm{U}(n))$.
- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ Cartan decomposition, θ Cartan involution.
- $H^{\mathbb{C}} \curvearrowright \mathfrak{m}^{\mathbb{C}}$ isotropy representation $(\mathrm{O}(n) \curvearrowright \mathfrak{sym}(n, \mathbb{R}))$.

Definition

A G-Higgs bundle on X is a pair (E, ϕ)

G-Higgs bundles

- X connected smooth projective curve/$\mathbb{C}$, $g(X) \geq 2$.
- G real reductive Lie group $(\mathrm{GL}(n,\mathbb{R}),\ \mathrm{U}(n))$.
- $H \leq G$ maximal compact subgroup $(\mathrm{O}(n),\ \mathrm{U}(n))$.
- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ Cartan decomposition, θ Cartan involution.
- $H^{\mathbb{C}} \curvearrowright \mathfrak{m}^{\mathbb{C}}$ isotropy representation $(\mathrm{O}(n) \curvearrowright \mathfrak{sym}(n,\mathbb{R}))$.

Definition

A G-Higgs bundle on X is a pair (E,ϕ) with $E \to X$ a holomorphic principal $H^{\mathbb{C}}$-bundle

G-Higgs bundles

- X connected smooth projective curve/$\mathbb{C}$, $g(X) \geq 2$.
- G real reductive Lie group $(\mathrm{GL}(n, \mathbb{R}),\ \mathrm{U}(n))$.
- $H \leq G$ maximal compact subgroup $(\mathrm{O}(n),\ \mathrm{U}(n))$.
- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ Cartan decomposition, θ Cartan involution.
- $H^{\mathbb{C}} \curvearrowright \mathfrak{m}^{\mathbb{C}}$ isotropy representation $(\mathrm{O}(n) \curvearrowright \mathfrak{sym}(n, \mathbb{R}))$.

Definition

A G-Higgs bundle on X is a pair (E, ϕ) with $E \to X$ a holomorphic principal $H^{\mathbb{C}}$-bundle and $\phi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$ the Higgs field.

G-Higgs bundles

- X connected smooth projective curve/$\mathbb{C}$, $g(X) \geq 2$.
- G real reductive Lie group $(\mathrm{GL}(n, \mathbb{R}),\ \mathrm{U}(n))$.
- $H \leq G$ maximal compact subgroup $(\mathrm{O}(n),\ \mathrm{U}(n))$.
- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ Cartan decomposition, θ Cartan involution.
- $H^{\mathbb{C}} \curvearrowright \mathfrak{m}^{\mathbb{C}}$ isotropy representation $(\mathrm{O}(n) \curvearrowright \mathfrak{sym}(n, \mathbb{R}))$.

Definition

A G-Higgs bundle on X is a pair (E, ϕ) with $E \to X$ a holomorphic principal $H^{\mathbb{C}}$-bundle and $\phi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$ the Higgs field.

Examples

G-Higgs bundles

- X connected smooth projective curve/$\mathbb{C}$, $g(X) \geq 2$.
- G real reductive Lie group $(\mathrm{GL}(n,\mathbb{R}),\ \mathrm{U}(n))$.
- $H \leq G$ maximal compact subgroup $(\mathrm{O}(n),\ \mathrm{U}(n))$.
- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ Cartan decomposition, θ Cartan involution.
- $H^{\mathbb{C}} \curvearrowright \mathfrak{m}^{\mathbb{C}}$ isotropy representation $(\mathrm{O}(n) \curvearrowright \mathfrak{sym}(n,\mathbb{R}))$.

Definition

A G-Higgs bundle on X is a pair (E,ϕ) with $E \to X$ a holomorphic principal $H^{\mathbb{C}}$-bundle and $\phi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$ the Higgs field.

Examples

1. $\mathrm{GL}(n,\mathbb{C})_{\mathbb{R}}$: E rk n vector bundle, $\phi : E \to E \otimes K$.

G-Higgs bundles

- X connected smooth projective curve/$\mathbb{C}$, $g(X) \geq 2$.
- G real reductive Lie group $(\mathrm{GL}(n,\mathbb{R}),\ \mathrm{U}(n))$.
- $H \leq G$ maximal compact subgroup $(\mathrm{O}(n),\ \mathrm{U}(n))$.
- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ Cartan decomposition, θ Cartan involution.
- $H^{\mathbb{C}} \curvearrowright \mathfrak{m}^{\mathbb{C}}$ isotropy representation $(\mathrm{O}(n) \curvearrowright \mathfrak{sym}(n,\mathbb{R}))$.

Definition

A G-Higgs bundle on X is a pair (E,ϕ) with $E \to X$ a holomorphic principal $H^{\mathbb{C}}$-bundle and $\phi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$ the Higgs field.

Examples

1. $\mathrm{GL}(n,\mathbb{C})_{\mathbb{R}}$: E rk n vector bundle, $\phi : E \to E \otimes K$.
2. $\mathrm{GL}(n,\mathbb{R})$: $E \cong E^*$ rk n v.b.+symmetric form, ϕ symmetric.

G-Higgs bundles

- X connected smooth projective curve/$\mathbb{C}$, $g(X) \geq 2$.
- G real reductive Lie group $(\mathrm{GL}(n, \mathbb{R}),\ \mathrm{U}(n))$.
- $H \leq G$ maximal compact subgroup $(\mathrm{O}(n),\ \mathrm{U}(n))$.
- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ Cartan decomposition, θ Cartan involution.
- $H^{\mathbb{C}} \curvearrowright \mathfrak{m}^{\mathbb{C}}$ isotropy representation $(\mathrm{O}(n) \curvearrowright \mathfrak{sym}(n, \mathbb{R}))$.

Definition

A G-Higgs bundle on X is a pair (E, ϕ) with $E \to X$ a holomorphic principal $H^{\mathbb{C}}$-bundle and $\phi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$ the Higgs field.

Examples

1. $\mathrm{GL}(n, \mathbb{C})_{\mathbb{R}}$: E rk n vector bundle, $\phi : E \to E \otimes K$.
2. $\mathrm{GL}(n, \mathbb{R})$: $E \cong E^*$ rk n v.b.+symmetric form, ϕ symmetric.
3. $\mathrm{U}(p, q)$: $E = V \oplus W$, $\phi = (\beta, \gamma) : V \overset{\gamma}{\mapsto} W \otimes K,\ W \overset{\beta}{\mapsto} V \otimes K$.

The Hitchin map

The Hitchin map

Definition

The Hitchin map is a morphism

$$h_G : \mathcal{M}(G) \quad \to \quad B_G := H^0(X, \oplus_i K^{d_i})$$

The Hitchin map

Definition

The Hitchin map is a morphism

$$\begin{array}{rcl} h_G : \mathcal{M}(G) & \to & B_G := H^0(X, \oplus_i K^{d_i}) \\ (E, \phi) & \mapsto & (p_1(\phi), \dots, p_r(\phi)) \end{array}$$

The Hitchin map

Definition

The Hitchin map is a morphism

$$\begin{array}{rcl} h_G : \mathcal{M}(G) & \rightarrow & B_G := H^0(X, \oplus_i K^{d_i}) \\ (E, \phi) & \mapsto & (p_1(\phi), \ldots, p_r(\phi)) \end{array}$$

$\mathcal{M}(G)$ *moduli space of ps. Higgs bundles*

The Hitchin map

Definition

The Hitchin map is a morphism

$$\begin{array}{rcl} h_G : \mathcal{M}(G) & \to & B_G := H^0(X, \oplus_i K^{d_i}) \\ (E, \phi) & \mapsto & (p_1(\phi), \dots, p_r(\phi)) \end{array}$$

$\mathcal{M}(G)$ *moduli space of ps. Higgs bundles*($\cong Hom(\pi_1, G)//G$)

The Hitchin map

Definition

The Hitchin map is a morphism

$$\begin{array}{rcl} h_G : \mathcal{M}(G) & \to & B_G := H^0(X, \oplus_i K^{d_i}) \\ (E,\phi) & \mapsto & (p_1(\phi), \ldots, p_r(\phi)) \end{array}$$

$\mathcal{M}(G)$ *moduli space of ps. Higgs bundles*($\cong Hom(\pi_1, G)//G$)
$r = rk_{\mathbb{R}} G$, $p_1, \ldots, p_r \in \mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{H^{\mathbb{C}}}$ *generators,*

The Hitchin map

Definition

The Hitchin map is a morphism

$$\begin{array}{rcl} h_G : \mathcal{M}(G) & \to & B_G := H^0(X, \oplus_i K^{d_i}) \\ (E,\phi) & \mapsto & (p_1(\phi), \dots, p_r(\phi)) \end{array}$$

$\mathcal{M}(G)$ *moduli space of ps. Higgs bundles*($\cong Hom(\pi_1, G)//G$)
$r = rk_{\mathbb{R}} G$, $p_1, \dots, p_r \in \mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{H^{\mathbb{C}}}$ *generators,* $d_i = \deg p_i$.

The Hitchin map

Definition

The Hitchin map is a morphism

$$\begin{array}{rcl} h_G : \mathcal{M}(G) & \to & B_G := H^0(X, \oplus_i K^{d_i}) \\ (E,\phi) & \mapsto & (p_1(\phi), \dots, p_r(\phi)) \end{array}$$

$\mathcal{M}(G)$ *moduli space of ps. Higgs bundles*($\cong Hom(\pi_1, G)//G$)
$r = rk_{\mathbb{R}} G$, $p_1, \dots, p_r \in \mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{H^{\mathbb{C}}}$ *generators,* $d_i = \deg p_i$.

Examples

The Hitchin map

Definition

The Hitchin map is a morphism

$$\begin{array}{rcl} h_G : \mathcal{M}(G) & \to & B_G := H^0(X, \oplus_i K^{d_i}) \\ (E, \phi) & \mapsto & (p_1(\phi), \dots, p_r(\phi)) \end{array}$$

$\mathcal{M}(G)$ *moduli space of ps. Higgs bundles*($\cong Hom(\pi_1, G)//G$)
$r = rk_{\mathbb{R}} G$, $p_1, \dots, p_r \in \mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{H^{\mathbb{C}}}$ *generators,* $d_i = \deg p_i$.

Examples

1. $\mathrm{GL}(n, \mathbb{C})_{\mathbb{R}}$ $p_i(x) = tr(x^i)$ $i = 1, \dots, n$.

The Hitchin map

Definition

The Hitchin map is a morphism

$$\begin{array}{rcl} h_G : \mathcal{M}(G) & \to & B_G := H^0(X, \oplus_i K^{d_i}) \\ (E, \phi) & \mapsto & (p_1(\phi), \dots, p_r(\phi)) \end{array}$$

$\mathcal{M}(G)$ *moduli space of ps. Higgs bundles*($\cong Hom(\pi_1, G)//G$)
$r = rk_{\mathbb{R}} G$, $p_1, \dots, p_r \in \mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{H^{\mathbb{C}}}$ *generators,* $d_i = \deg p_i$.

Examples

1. $\mathrm{GL}(n, \mathbb{C})_{\mathbb{R}}$ $p_i(x) = tr(x^i)$ $i = 1, \dots, n$.
2. $\mathrm{GL}(n, \mathbb{R})$: $p_i(x) = tr(x^i)$, $i = 1, \dots, n$ (split).

The Hitchin map

Definition

The Hitchin map is a morphism

$$\begin{array}{rcl} h_G : \mathcal{M}(G) & \to & B_G := H^0(X, \oplus_i K^{d_i}) \\ (E, \phi) & \mapsto & (p_1(\phi), \ldots, p_r(\phi)) \end{array}$$

$\mathcal{M}(G)$ *moduli space of ps. Higgs bundles*($\cong Hom(\pi_1, G)//G$)
$r = rk_{\mathbb{R}} G$, $p_1, \ldots, p_r \in \mathbb{C}[\mathfrak{m}^{\mathbb{C}}]^{H^{\mathbb{C}}}$ *generators*, $d_i = \deg p_i$.

Examples

1. $\mathrm{GL}(n, \mathbb{C})_{\mathbb{R}}$ $p_i(x) = tr(x^i)$ $i = 1, \ldots, n$.
2. $\mathrm{GL}(n, \mathbb{R})$: $p_i(x) = tr(x^i)$, $i = 1, \ldots, n$ (split).
3. $\mathrm{U}(p, q)$: $p_i(x) = tr(x^{2i}), i = 1, \ldots, q$ $(p > q)$.

Understanding the Hitchin map

Understanding the Hitchin map

Two major steps:

Understanding the Hitchin map

Two major steps:

1. Study of the fibers

Understanding the Hitchin map

Two major steps:

1. Study of the fibers $\rightsquigarrow$ abelianization (complex groups).

Understanding the Hitchin map

Two major steps:

1. Study of the fibers$\rightsquigarrow$ abelianization (complex groups).
2. Existence of a section

Understanding the Hitchin map

Two major steps:

1. Study of the fibers$\rightsquigarrow$ abelianization (complex groups).
2. Existence of a section$\rightsquigarrow$ Hitchin section ($\mathbb{C}$/split groups)

Understanding the Hitchin map

Two major steps:

1. Study of the fibers $\rightsquigarrow$ abelianization (complex groups).
2. Existence of a section $\rightsquigarrow$ Hitchin section ($\mathbb{C}$/split groups),
   Hitchin–Kostant–Rallis section
   (real groups).

Abelianization

Abelianization

- **Idea:** Higgs bundles $\leftrightarrow$ eigen-line bundles

Abelianization

- <u>**Idea:**</u> Higgs bundles $\leftrightarrow$ eigen-line bundles
- Eigenvalues parametrized by

Abelianization

- **Idea:** Higgs bundles $\leftrightarrow$ eigen-line bundles
- Eigenvalues parametrized by
  (a) characteristic polynomials (unordered eigenvalues)

Abelianization

- **Idea:** Higgs bundles $\leftrightarrow$ eigen-line bundles
- Eigenvalues parametrized by
  (a) characteristic polynomials (unordered eigenvalues)$\rightsquigarrow$ spectral curve,

Abelianization

- **Idea:** Higgs bundles $\leftrightarrow$ eigen-line bundles
- Eigenvalues parametrized by
  (a) characteristic polynomials (unordered eigenvalues)$\rightsquigarrow$ spectral curve,
  (b) diagonal matrices (ordered eigenvalues)

Abelianization

- **Idea:** Higgs bundles $\leftrightarrow$ eigen-line bundles
- Eigenvalues parametrized by
  (a) characteristic polynomials (unordered eigenvalues)$\rightsquigarrow$ spectral curve,
  (b) diagonal matrices (ordered eigenvalues)$\rightsquigarrow$ cameral curve.

Abelianization

- **Idea:** Higgs bundles $\leftrightarrow$ eigen-line bundles
- Eigenvalues parametrized by
  (a) characteristic polynomials (unordered eigenvalues)$\rightsquigarrow$ spectral curve,
  (b) diagonal matrices (ordered eigenvalues)$\rightsquigarrow$ cameral curve.
- (b) generalises to all reductive groups:

Abelianization

- **<u>Idea:</u>** Higgs bundles $\leftrightarrow$ eigen-line bundles
- Eigenvalues parametrized by
  (a) characteristic polynomials (unordered eigenvalues)$\rightsquigarrow$ spectral curve,
  (b) diagonal matrices (ordered eigenvalues)$\rightsquigarrow$ cameral curve.
- (b) generalises to all reductive groups:
  diagonal matrices $\leftrightarrow$ Cartan subalgebras $\mathfrak{d}^{\mathbb{C}}$

Abelianization

- <u>**Idea:**</u> Higgs bundles $\leftrightarrow$ eigen-line bundles
- Eigenvalues parametrized by
  (a) characteristic polynomials (unordered eigenvalues)$\rightsquigarrow$ spectral curve,
  (b) diagonal matrices (ordered eigenvalues)$\rightsquigarrow$ cameral curve.
- (b) generalises to all reductive groups:
  diagonal matrices $\leftrightarrow$ Cartan subalgebras $\mathfrak{d}^{\mathbb{C}}$
- eigenbundles $\leftrightarrow$ principal bundles of tori $D^{\mathbb{C}}$.

Spectral vs. cameral covers

Spectral vs. cameral covers

(a) **Spectral techniques** $(\mathrm{GL}(2,\mathbb{C}))$:

Spectral vs. cameral covers

(a) **Spectral techniques** $(\mathrm{GL}(2,\mathbb{C}))$:

- $(E,\phi) \rightsquigarrow char(\phi)=\lambda^2+\pi^*a_1\lambda+\pi^*a_2 =: s_{\overline{a}} \in H^0(|K|,\pi^*K)$

Spectral vs. cameral covers

(a) **Spectral techniques** $(\mathrm{GL}(2,\mathbb{C}))$:

- $(E,\phi) \rightsquigarrow char(\phi) = \lambda^2 + \pi^* a_1\lambda + \pi^* a_2 =: s_{\overline{a}} \in H^0(|K|, \pi^* K)$
- $\widetilde{X}_{\overline{a}} := \{s_{\overline{a}} = 0\}$ spectral curve

Spectral vs. cameral covers

(a) Spectral techniques $(\mathrm{GL}(2,\mathbb{C}))$:

- $(E,\phi) \rightsquigarrow char(\phi) = \lambda^2 + \pi^* a_1 \lambda + \pi^* a_2 =: s_{\overline{a}} \in H^0(|K|, \pi^* K)$
- $\widetilde{X}_{\overline{a}} := \{s_{\overline{a}} = 0\}$ spectral curve $ker(\pi^*\phi - \lambda Id) = L_\lambda$.

Spectral vs. cameral covers

(a) **Spectral techniques** $(\mathrm{GL}(2,\mathbb{C}))$:

- $(E,\phi) \rightsquigarrow char(\phi) = \lambda^2 + \pi^* a_1 \lambda + \pi^* a_2 =: s_{\overline{a}} \in H^0(|K|, \pi^* K)$
- $\widetilde{X}_{\overline{a}} := \{s_{\overline{a}} = 0\}$ spectral curve $ker(\pi^*\phi - \lambda Id) = L_\lambda$.

Theorem (Hitchin, 87)

$(E,\phi) \leftrightarrow L_\lambda$ yields an isomorphism $h^{-1}(\overline{a}) \cong Pic(\widetilde{X}_{\overline{a}})$

Spectral vs. cameral covers

(a) Spectral techniques $(\mathrm{GL}(2,\mathbb{C}))$:

- $(E,\phi) \rightsquigarrow char(\phi) = \lambda^2 + \pi^* a_1 \lambda + \pi^* a_2 =: s_{\overline{a}} \in H^0(|K|, \pi^* K)$
- $\widetilde{X}_{\overline{a}} := \{s_{\overline{a}} = 0\}$ spectral curve $ker(\pi^*\phi - \lambda Id) = L_\lambda$.

Theorem (Hitchin, 87)

$(E,\phi) \leftrightarrow L_\lambda$ yields an isomorphism $h^{-1}(\overline{a}) \cong Pic(\widetilde{X}_{\overline{a}})$

(b) Cameral techniques$(\mathrm{GL}(2,\mathbb{C}))$

Spectral vs. cameral covers

(a) Spectral techniques $(\mathrm{GL}(2,\mathbb{C}))$:

- $(E,\phi) \rightsquigarrow char(\phi) = \lambda^2 + \pi^* a_1 \lambda + \pi^* a_2 =: s_{\overline{a}} \in H^0(|K|, \pi^* K)$
- $\widetilde{X}_{\overline{a}} := \{s_{\overline{a}} = 0\}$ spectral curve $ker(\pi^*\phi - \lambda Id) = L_\lambda$.

Theorem (Hitchin, 87)

$(E,\phi) \leftrightarrow L_\lambda$ yields an isomorphism $h^{-1}(\overline{a}) \cong Pic(\widetilde{X}_{\overline{a}})$

(b) Cameral techniques$(\mathrm{GL}(2,\mathbb{C}))$

- $(E,\phi) \rightsquigarrow \phi_{ss}$ semisimple part

Spectral vs. cameral covers

(a) Spectral techniques $(\mathrm{GL}(2,\mathbb{C}))$:

- $(E,\phi) \rightsquigarrow char(\phi) = \lambda^2 + \pi^* a_1\lambda + \pi^* a_2 =: s_{\overline{a}} \in H^0(|K|, \pi^* K)$
- $\widetilde{X}_{\overline{a}} := \{s_{\overline{a}} = 0\}$ spectral curve $ker(\pi^*\phi - \lambda Id) = L_\lambda$.

Theorem (Hitchin, 87)

$(E,\phi) \leftrightarrow L_\lambda$ yields an isomorphism $h^{-1}(\overline{a}) \cong Pic(\widetilde{X}_{\overline{a}})$

(b) Cameral techniques$(\mathrm{GL}(2,\mathbb{C}))$

- $(E,\phi) \rightsquigarrow \phi_{ss}$ semisimple part up to conjugation $\phi_{ss} \in K \oplus K$.

Spectral vs. cameral covers

(a) **Spectral techniques** $(\mathrm{GL}(2,\mathbb{C}))$:

- $(E,\phi) \rightsquigarrow char(\phi) = \lambda^2 + \pi^* a_1 \lambda + \pi^* a_2 =: s_{\overline{a}} \in H^0(|K|, \pi^* K)$
- $\widetilde{X}_{\overline{a}} := \{s_{\overline{a}} = 0\}$ spectral curve $ker(\pi^*\phi - \lambda Id) = L_\lambda$.

Theorem (Hitchin, 87)

$(E,\phi) \leftrightarrow L_\lambda$ yields an isomorphism $h^{-1}(\overline{a}) \cong Pic(\widetilde{X}_{\overline{a}})$

(b) **Cameral techniques** $(\mathrm{GL}(2,\mathbb{C}))$

- $(E,\phi) \rightsquigarrow \phi_{ss}$ semisimple part up to conjugation $\phi_{ss} \in K \oplus K$.
- $\overline{a} := h(E,\phi) = (tr\phi, \det\phi) : X \to K \oplus K^2$

Spectral vs. cameral covers

(a) **Spectral techniques** $(\mathrm{GL}(2,\mathbb{C}))$:

- $(E,\phi) \rightsquigarrow char(\phi) = \lambda^2 + \pi^* a_1 \lambda + \pi^* a_2 =: s_{\overline{a}} \in H^0(|K|, \pi^* K)$
- $\widetilde{X}_{\overline{a}} := \{s_{\overline{a}} = 0\}$ spectral curve $ker(\pi^*\phi - \lambda Id) = L_\lambda$.

Theorem (Hitchin, 87)

$(E,\phi) \leftrightarrow L_\lambda$ yields an isomorphism $h^{-1}(\overline{a}) \cong Pic(\widetilde{X}_{\overline{a}})$

(b) **Cameral techniques** $(\mathrm{GL}(2,\mathbb{C}))$

- $(E,\phi) \rightsquigarrow \phi_{ss}$ semisimple part up to conjugation $\phi_{ss} \in K \oplus K$.
- $\overline{a} := h(E,\phi) = (tr\phi, \det\phi) : X \to K \oplus K^2$
- $K \oplus K \to K \oplus K^2$

Spectral vs. cameral covers

(a) **Spectral techniques** ($\mathrm{GL}(2,\mathbb{C})$):

- $(E,\phi) \rightsquigarrow char(\phi) = \lambda^2 + \pi^* a_1\lambda + \pi^* a_2 =: s_{\overline{a}} \in H^0(|K|, \pi^*K)$
- $\widetilde{X}_{\overline{a}} := \{s_{\overline{a}} = 0\}$ spectral curve $ker(\pi^*\phi - \lambda Id) = L_\lambda$.

Theorem (Hitchin, 87)

$(E,\phi) \leftrightarrow L_\lambda$ yields an isomorphism $h^{-1}(\overline{a}) \cong Pic(\widetilde{X}_{\overline{a}})$

(b) **Cameral techniques**($\mathrm{GL}(2,\mathbb{C})$)

- $(E,\phi) \rightsquigarrow \phi_{ss}$ semisimple part up to conjugation $\phi_{ss} \in K \oplus K$.
- $\overline{a} := h(E,\phi) = (tr\phi, \det\phi) : X \to K \oplus K^2$
- $K \oplus K \to K \oplus K^2$ $(l, l') \mapsto (l + l', l \cdot l')$

Spectral vs. cameral covers

(a) Spectral techniques $(\mathrm{GL}(2,\mathbb{C}))$:

- $(E,\phi) \rightsquigarrow char(\phi) = \lambda^2 + \pi^* a_1 \lambda + \pi^* a_2 =: s_{\overline{a}} \in H^0(|K|, \pi^* K)$
- $\widetilde{X}_{\overline{a}} := \{s_{\overline{a}} = 0\}$ spectral curve $ker(\pi^*\phi - \lambda Id) = L_\lambda$.

Theorem (Hitchin, 87)

$(E,\phi) \leftrightarrow L_\lambda$ yields an isomorphism $h^{-1}(\overline{a}) \cong Pic(\widetilde{X}_{\overline{a}})$

(b) Cameral techniques$(\mathrm{GL}(2,\mathbb{C}))$

- $(E,\phi) \rightsquigarrow \phi_{ss}$ semisimple part up to conjugation $\phi_{ss} \in K \oplus K$.
- $\overline{a} := h(E,\phi) = (tr\phi, \det\phi) : X \to K \oplus K^2$
- $K \oplus K \to K \oplus K^2$ $(l, l') \mapsto (l + l', l \cdot l')$

Theorem (Donagi, 93, D-Gaitsgory 02)

$h^{-1}(\overline{a}) \cong H^1(\widehat{X}_{\overline{a}}, (\mathbb{C}^\times)^2)^{S_2}$ where $\widehat{X}_{\overline{a}} := \overline{a}^* K^{\oplus 2}$ cameral cover

Quasi-split real groups

Quasi-split real groups

Examples:

- complex groups,

Quasi-split real groups

Examples:

- complex groups,
- split $(\mathrm{GL}(n, \mathbb{R}),\ \mathrm{Sp}(2n, \mathbb{R}),\ \mathrm{SO}(n, n+1),\ \mathrm{SO}(n, n))$

Quasi-split real groups

Examples:

- complex groups,
- split $(\mathrm{GL}(n,\mathbb{R}),\ \mathrm{Sp}(2n,\mathbb{R}),\ \mathrm{SO}(n,n+1),\ \mathrm{SO}(n,n))$
- $\mathrm{SU}(p,p),\ \mathrm{SU}(p+1,p),\ \mathrm{SO}(p,p+2)$

Quasi-split real groups

Examples:

- complex groups,
- split ($\mathrm{GL}(n,\mathbb{R})$, $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{SO}(n,n+1)$, $\mathrm{SO}(n,n)$)
- $\mathrm{SU}(p,p)$, $\mathrm{SU}(p+1,p)$, $\mathrm{SO}(p,p+2)$

What is special about them?

Quasi-split real groups

Examples:

- complex groups,
- split ($\mathrm{GL}(n,\mathbb{R})$, $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{SO}(n,n+1)$, $\mathrm{SO}(n,n)$)
- $\mathrm{SU}(p,p)$, $\mathrm{SU}(p+1,p)$, $\mathrm{SO}(p,p+2)$

What is special about them?

- Most elements in $\mathfrak{m}^{\mathbb{C}}$ have one dimensional eigenspaces (matrix groups).

Quasi-split real groups

Examples:

- complex groups,
- split $(\mathrm{GL}(n,\mathbb{R}), \mathrm{Sp}(2n,\mathbb{R}), \mathrm{SO}(n,n+1), \mathrm{SO}(n,n))$
- $\mathrm{SU}(p,p), \mathrm{SU}(p+1,p), \mathrm{SO}(p,p+2)$

What is special about them?

- Most elements in $\mathfrak{m}^{\mathbb{C}}$ have one dimensional eigenspaces (matrix groups).
- In other words $\mathfrak{m}_{reg} \subset \mathfrak{g}_{reg}$ (any group).

The Hitchin fibers for quasi-split real groups

- G quasi-split real reductive group; θ Cartan involution.

The Hitchin fibers for quasi-split real groups

- G quasi-split real reductive group; θ Cartan involution.
- $\mathfrak{d}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ Cartan subalgebra

The Hitchin fibers for quasi-split real groups

- G quasi-split real reductive group; θ Cartan involution.
- $\mathfrak{d}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ θ-inv. Cartan subalgebra

The Hitchin fibers for quasi-split real groups

- G quasi-split real reductive group; θ Cartan involution.
- $\mathfrak{d}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ θ-inv. Cartan subalgebra $\mathfrak{d}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{a}^{\mathbb{C}}$; $\mathfrak{a}^{\mathbb{C}} \subset \mathfrak{m}^{\mathbb{C}}$.

The Hitchin fibers for quasi-split real groups

- G quasi-split real reductive group; θ Cartan involution.
- $\mathfrak{d}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ θ-inv. Cartan subalgebra $\mathfrak{d}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{a}^{\mathbb{C}}$; $\mathfrak{a}^{\mathbb{C}} \subset \mathfrak{m}^{\mathbb{C}}$.
- W Weyl group

The Hitchin fibers for quasi-split real groups

- G quasi-split real reductive group; θ Cartan involution.
- $\mathfrak{d}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ θ-inv. Cartan subalgebra $\mathfrak{d}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{a}^{\mathbb{C}}$; $\mathfrak{a}^{\mathbb{C}} \subset \mathfrak{m}^{\mathbb{C}}$.
- W Weyl group
- We note $\oplus_i K^{d_i} = \mathfrak{a}^{\mathbb{C}} \otimes K/W(\mathfrak{a}) \subset \mathfrak{d}^{\mathbb{C}} \otimes K/W$

The Hitchin fibers for quasi-split real groups

- G quasi-split real reductive group; θ Cartan involution.
- $\mathfrak{d}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ θ-inv. Cartan subalgebra $\mathfrak{d}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{a}^{\mathbb{C}}$; $\mathfrak{a}^{\mathbb{C}} \subset \mathfrak{m}^{\mathbb{C}}$.
- W Weyl group
- We note $\oplus_i K^{d_i} = \mathfrak{a}^{\mathbb{C}} \otimes K/W(\mathfrak{a}) \subset \mathfrak{d}^{\mathbb{C}} \otimes K/W$

Theorem (García-Prada, P-N.)

Let $a \in B_G$, define its associated cameral cover by

The Hitchin fibers for quasi-split real groups

- G quasi-split real reductive group; θ Cartan involution.
- $\mathfrak{d}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ θ-inv. Cartan subalgebra $\mathfrak{d}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{a}^{\mathbb{C}}$; $\mathfrak{a}^{\mathbb{C}} \subset \mathfrak{m}^{\mathbb{C}}$.
- W Weyl group
- We note $\oplus_i K^{d_i} = \mathfrak{a}^{\mathbb{C}} \otimes K/W(\mathfrak{a}) \subset \mathfrak{d}^{\mathbb{C}} \otimes K/W$

Theorem (García-Prada, P-N.)

Let $a \in B_G$, define its associated cameral cover by

$$\begin{array}{c} \mathfrak{d}^{\mathbb{C}} \otimes K \\ \downarrow \\ \mathfrak{d}^{\mathbb{C}} \otimes K/W \end{array}$$

The Hitchin fibers for quasi-split real groups

- G quasi-split real reductive group; θ Cartan involution.
- $\mathfrak{d}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ θ-inv. Cartan subalgebra $\mathfrak{d}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{a}^{\mathbb{C}}$; $\mathfrak{a}^{\mathbb{C}} \subset \mathfrak{m}^{\mathbb{C}}$.
- W Weyl group
- We note $\oplus_i K^{d_i} = \mathfrak{a}^{\mathbb{C}} \otimes K/W(\mathfrak{a}) \subset \mathfrak{d}^{\mathbb{C}} \otimes K/W$

Theorem (García-Prada, P-N.)

Let $a \in B_G$, define its associated cameral cover by

$$\begin{array}{ccc} & & \mathfrak{d}^{\mathbb{C}} \otimes K \\ & & \downarrow \\ X & \xrightarrow[a]{} & \mathfrak{d}^{\mathbb{C}} \otimes K/W \end{array}$$

The Hitchin fibers for quasi-split real groups

- G quasi-split real reductive group; θ Cartan involution.
- $\mathfrak{d}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ θ-inv. Cartan subalgebra $\mathfrak{d}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{a}^{\mathbb{C}}$; $\mathfrak{a}^{\mathbb{C}} \subset \mathfrak{m}^{\mathbb{C}}$.
- W Weyl group
- We note $\oplus_i K^{d_i} = \mathfrak{a}^{\mathbb{C}} \otimes K/W(\mathfrak{a}) \subset \mathfrak{d}^{\mathbb{C}} \otimes K/W$

Theorem (García-Prada, P-N.)

Let $a \in B_G$, define its associated cameral cover by

$$\begin{array}{ccc} \widehat{X}_a & \longrightarrow & \mathfrak{d}^{\mathbb{C}} \otimes K \\ \downarrow & & \downarrow \\ X & \xrightarrow[a]{} & \mathfrak{d}^{\mathbb{C}} \otimes K/W \end{array}$$

The Hitchin fibers for quasi-split real groups

- G quasi-split real reductive group; θ Cartan involution.
- $\mathfrak{d}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ θ-inv. Cartan subalgebra $\mathfrak{d}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{a}^{\mathbb{C}}$; $\mathfrak{a}^{\mathbb{C}} \subset \mathfrak{m}^{\mathbb{C}}$.
- W Weyl group
- We note $\oplus_i K^{d_i} = \mathfrak{a}^{\mathbb{C}} \otimes K/W(\mathfrak{a}) \subset \mathfrak{d}^{\mathbb{C}} \otimes K/W$

Theorem (García-Prada, P-N.)

Let $a \in B_G$, define its associated cameral cover by

$$\begin{array}{ccc} \widehat{X}_a & \longrightarrow & \mathfrak{d}^{\mathbb{C}} \otimes K \\ \downarrow & & \downarrow \\ X & \xrightarrow[a]{} & \mathfrak{d}^{\mathbb{C}} \otimes K/W \end{array}$$

Then $h^{-1}(a) \cong H^1(\widehat{X}_a, D^{\mathbb{C}})^{W,\theta}$

The Hitchin fibers for quasi-split real groups

- G quasi-split real reductive group; θ Cartan involution.
- $\mathfrak{d}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ θ-inv. Cartan subalgebra $\mathfrak{d}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{a}^{\mathbb{C}}$; $\mathfrak{a}^{\mathbb{C}} \subset \mathfrak{m}^{\mathbb{C}}$.
- W Weyl group
- We note $\oplus_i K^{d_i} = \mathfrak{a}^{\mathbb{C}} \otimes K/W(\mathfrak{a}) \subset \mathfrak{d}^{\mathbb{C}} \otimes K/W$

Theorem (García-Prada, P-N.)

Let $a \in B_G$, define its associated cameral cover by

$$\begin{array}{ccc} \widehat{X}_a & \longrightarrow & \mathfrak{d}^{\mathbb{C}} \otimes K \\ \downarrow & & \downarrow \\ X & \xrightarrow[a]{} & \mathfrak{d}^{\mathbb{C}} \otimes K/W \end{array}$$

Then $h^{-1}(a) \cong H^1(\widehat{X}_a, D^{\mathbb{C}})^{W,\theta}$ (regular)

Two applications

Two applications

Corollary

If $G < G^{\mathbb{C}}$ *is split then* $h_G^{-1}(\overline{a}) = h_{G^{\mathbb{C}}}^{-1}(a)[2]$.

Two applications

Corollary

If $G < G^{\mathbb{C}}$ *is split then* $h_G^{-1}(\overline{a}) = h_{G^{\mathbb{C}}}^{-1}(a)[2]$.

Theorem (P-N.)

If $G = SU(p+1, p)$, $\overline{a} \in B_{SU(p+1,p)}$ *generic*

Two applications

Corollary

If $G < G^{\mathbb{C}}$ is split then $h_G^{-1}(\overline{a}) = h_{G^{\mathbb{C}}}^{-1}(a)[2]$.

Theorem (P-N.)

If $G = SU(p+1,p)$, $\overline{a} \in B_{SU(p+1,p)}$ generic $h_G^{-1}(a)$ is a bundle of projective spaces over $Pic(Y)$

Two applications

Corollary

If $G < G^{\mathbb{C}}$ is split then $h_G^{-1}(\overline{a}) = h_{G^{\mathbb{C}}}^{-1}(a)[2]$.

Theorem (P-N.)

If $G = SU(p+1,p)$, $\overline{a} \in B_{SU(p+1,p)}$ generic $h_G^{-1}(a)$ is a bundle of projective spaces over $Pic(Y)$
$Y = \{\lambda^{2p} + a_2\lambda^{2p-2} + \cdots + a_{2p} = 0\}$.

The Hitchin–Kostant–Rallis section

The Hitchin–Kostant–Rallis section

Theorem (García-Prada, P-N., Ramanan)

Let G be quasi-split. Then, there exists a section

$$s : B_G \to \mathcal{M}(G)^{smooth}$$

such that $s(a) = (E_a, \phi_a)$ is everywhere regular ($\phi_a(x) \in \mathfrak{m}_{reg}$).

What's next?

What's next?

1. "Non-abelianizable" case.

What's next?

1. "Non-abelianizable" case.
   Sheaves of non abelian groups involved
   e.g. for $\mathrm{U}(p,q)$, $\mathrm{GL}(p-q,\mathbb{C})\times(\mathbb{C}^\times)^q$

What's next?

1. "Non-abelianizable" case.
   Sheaves of non abelian groups involved
   e.g. for $\mathrm{U}(p,q)$, $\mathrm{GL}(p-q,\mathbb{C})\times(\mathbb{C}^\times)^q$
   Decompose $\mathcal{M}(G)$ into abelian $\mathcal{M}(G_{qs})$ and non abelian M

What's next?

1. "Non-abelianizable" case.
   Sheaves of non abelian groups involved
   e.g. for $\mathrm{U}(p,q)$, $\mathrm{GL}(p-q,\mathbb{C})\times(\mathbb{C}^\times)^q$
   Decompose $\mathcal{M}(G)$ into abelian $\mathcal{M}(G_{qs})$ and non abelian M
   e.g. $h^{-1}_{U(p,q)}(a)$ in terms of $B\mathrm{GL}(p-q,\mathbb{C})$ and $h^{-1}_{U(p,p)}(a)$.

What's next?

1. "Non-abelianizable" case.
   Sheaves of non abelian groups involved
   e.g. for $\mathrm{U}(p,q)$, $\mathrm{GL}(p-q,\mathbb{C})\times(\mathbb{C}^\times)^q$
   Decompose $\mathcal{M}(G)$ into abelian $\mathcal{M}(G_{qs})$ and non abelian M
   e.g. $h^{-1}_{U(p,q)}(a)$ in terms of $B\mathrm{GL}(p-q,\mathbb{C})$ and $h^{-1}_{U(p,p)}(a)$.
2. Geometry of the HKR section.

What's next?

1. "Non-abelianizable" case.
   Sheaves of non abelian groups involved
   e.g. for $\mathrm{U}(p,q)$, $\mathrm{GL}(p-q,\mathbb{C})\times(\mathbb{C}^\times)^q$
   Decompose $\mathcal{M}(G)$ into abelian $\mathcal{M}(G_{qs})$ and non abelian M
   e.g. $h^{-1}_{U(p,q)}(a)$ in terms of $B\mathrm{GL}(p-q,\mathbb{C})$ and $h^{-1}_{U(p,p)}(a)$.
2. Geometry of the HKR section.
   Image consists of Anosov representations

What's next?

1. "Non-abelianizable" case.
   Sheaves of non abelian groups involved
   e.g. for $\mathrm{U}(p,q)$, $\mathrm{GL}(p-q,\mathbb{C})\times(\mathbb{C}^\times)^q$
   Decompose $\mathcal{M}(G)$ into abelian $\mathcal{M}(G_{qs})$ and non abelian M
   e.g. $h^{-1}_{U(p,q)}(a)$ in terms of $B\mathrm{GL}(p-q,\mathbb{C})$ and $h^{-1}_{U(p,p)}(a)$.
2. Geometry of the HKR section.
   Image consists of Anosov representations for minimal parabolic subgroups.

What's next?

1. "Non-abelianizable" case.
   Sheaves of non abelian groups involved
   e.g. for $\mathrm{U}(p,q)$, $\mathrm{GL}(p-q,\mathbb{C})\times(\mathbb{C}^\times)^q$
   Decompose $\mathcal{M}(G)$ into abelian $\mathcal{M}(G_{qs})$ and non abelian M
   e.g. $h^{-1}_{U(p,q)}(a)$ in terms of $B\mathrm{GL}(p-q,\mathbb{C})$ and $h^{-1}_{U(p,p)}(a)$.
2. Geometry of the HKR section.
   Image consists of Anosov representations for minimal parabolic subgroups.
3. Langlands duality

What's next?

1. "Non-abelianizable" case.
   Sheaves of non abelian groups involved
   e.g. for $\mathrm{U}(p,q)$, $\mathrm{GL}(p-q,\mathbb{C})\times(\mathbb{C}^\times)^q$
   Decompose $\mathcal{M}(G)$ into abelian $\mathcal{M}(G_{qs})$ and non abelian M
   e.g. $h^{-1}_{U(p,q)}(a)$ in terms of $B\mathrm{GL}(p-q,\mathbb{C})$ and $h^{-1}_{U(p,p)}(a)$.
2. Geometry of the HKR section.
   Image consists of Anosov representations for minimal parabolic subgroups.
3. Langlands duality
   Toy example: Hitchin ($\mathrm{U}(p,p)$ vs. $\mathrm{Sp}(2p,\mathbb{C})$-Higgs bundles)

What's next?

1. "Non-abelianizable" case.
   Sheaves of non abelian groups involved
   e.g. for $\mathrm{U}(p,q)$, $\mathrm{GL}(p-q,\mathbb{C})\times(\mathbb{C}^\times)^q$
   Decompose $\mathcal{M}(G)$ into abelian $\mathcal{M}(G_{qs})$ and non abelian M
   e.g. $h^{-1}_{U(p,q)}(a)$ in terms of $B\mathrm{GL}(p-q,\mathbb{C})$ and $h^{-1}_{U(p,p)}(a)$.
2. Geometry of the HKR section.
   Image consists of Anosov representations for minimal parabolic subgroups.
3. Langlands duality
   Toy example: Hitchin ($\mathrm{U}(p,p)$ vs. $\mathrm{Sp}(2p,\mathbb{C})$-Higgs bundles)