Monads over projective varieties

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Universidade de Évora

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Joint work with Simone Marchesi and Helena Soares

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The cohomology sheaf of the monad M is $E := \ker \beta / \mathrm{im}\, \alpha$.

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Questions

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   Given A, B, and C, what does the family of monads of type $M_\bullet$ look like?

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   Given a monad of type $M_\bullet$, what can we say about its cohomology sheaf? Is it simple? Is it stable?

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$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^a \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}{}^b \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^c \longrightarrow 0$$

if and only if one of the following holds:

1. $b \geq 2c + n$ and $b \geq a + c$;
2. $b \geq a + c + n$.

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Costa, Miró Roig 2009 Let $Q_n \subset \mathbb{P}^{n+1}$ be a quadric hypersurface, $n \geq 3$. There is a monad

$$0 \longrightarrow \mathcal{O}_{Q_n}(-1)^a \xrightarrow{\alpha} \mathcal{O}_{Q_n}{}^b \xrightarrow{\beta} \mathcal{O}_{Q_n}(1)^c \longrightarrow 0$$

if and only if one of the conditions 1 and 2 holds.

Theorem (Marchesi, Soares, __)

Let X be a projective variety of dimension n. Let L be a very ample line bundle over X such that the immersion $X \xrightarrow{|L|} \mathbb{P}^N$ is ACM. There is a monad

$$0 \longrightarrow (L^\vee)^a \xrightarrow{\alpha} {\mathcal{O}_X}^b \xrightarrow{\beta} L^c \longrightarrow 0$$

if and only if one of the following holds:

1. $b \geq 2c + n$ *and* $b \geq a + c$;
2. $b \geq a + c + n$.

Consider the Segre embedding $\mathbb{P}^l \times \mathbb{P}^m \longrightarrow \mathbb{P}^{lm+l+m}$, with homogeneous coordinates $[z_{00} : \cdots : z_{lm}]$ in $\mathbb{P}^{lm+l+m}$.

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Example over a Segre variety

Consider the Segre embedding $\mathbb{P}^l \times \mathbb{P}^m \longrightarrow \mathbb{P}^{lm+l+m}$, with homogeneous coordinates $[z_{00} : \cdots : z_{lm}]$ in $\mathbb{P}^{lm+l+m}$. The Segre variety is defined by the polynomials

$$z_{ab}z_{cd} - z_{ad}z_{cb},$$

with $0 \leq a, c \leq l$ and $0 \leq b, d \leq m$.

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$$z_{ab}z_{cd} = \frac{1}{4}\left((z_{ab} + z_{cd})^2 - (z_{ab} - z_{cd})^2\right) = \frac{1}{4}\left({u_{abcd}}^2 - {v_{abcd}}^2\right),$$

we get for any point in the Segre variety

$$ {u_{abcd}}^2 - {v_{abcd}}^2 - {u_{adcb}}^2 + {v_{adcb}}^2 = 0.$$

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$$q_{abcd} := (u_{abcd}, v_{abcd}, u_{adcb}, v_{adcb}),$$

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Choose linear forms $w_1, \ldots, w_{2e}$ that cannot vanish simultaneously on the Segre variety and set

$$A_1 = \begin{bmatrix} w_e & & \\ \vdots & \ddots & \\ w_1 & & w_e \\ & \ddots & \vdots \\ & & w_1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} w_{2e} & & \\ \vdots & \ddots & \\ w_{e+1} & & w_{2e} \\ & \ddots & \vdots \\ & & w_{e+1} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} w_1 & \cdots & w_e & & \\ & \ddots & & \ddots & \\ & & w_1 & \cdots & w_e \end{bmatrix}, \quad B_2 = \begin{bmatrix} w_{e+1} & \cdots & w_{2e} & & \\ & \ddots & & \ddots & \\ & & w_{e+1} & \cdots & w_{2e} \end{bmatrix},$$

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$$A = \begin{bmatrix} -A_2 \\ A_1 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 & B_2 \end{bmatrix},$$

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Example over a Segre variety

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^l\times\mathbb{P}^m}(-1,-1)^k \xrightarrow[\alpha]{} \mathcal{O}_{\mathbb{P}^l\times\mathbb{P}^m}^{2k+lm+l+m-1-p} \xrightarrow[\beta]{} \mathcal{O}_{\mathbb{P}^l\times\mathbb{P}^m}(1,1)^k \longrightarrow 0$$

Families of monads

What does the family of monads of type $M_\bullet$ look like?

$$M_\bullet : \quad 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

Families of monads

Theorem (Marchesi, Soares, __)

Let a, b, and c satisfy the conditions of the prevoius theorem, and suppose that $1 \leq c \leq 2$. Then for any surjective morphism $\beta \in Hom\big(\mathcal{O}_{\mathbb{P}^n}{}^b, \mathcal{O}_{\mathbb{P}^n}(1)^c\big)$ there is a morphism $\alpha \in Hom\big(\mathcal{O}_{\mathbb{P}^n}(-1)^a, \mathcal{O}_{\mathbb{P}^n}{}^b\big)$ yielding a monad

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^a \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}{}^b \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^c \longrightarrow 0.$$

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Theorem (Marchesi, Soares, __)

Let a, b, and c satisfy the conditions of the prevoius theorem, and suppose that $1 \leq c \leq 2$*. Then for any surjective morphism* $\beta \in Hom\big(\mathcal{O}_{\mathbb{P}^n}{}^b, \mathcal{O}_{\mathbb{P}^n}(1)^c\big)$ *there is a morphism* $\alpha \in Hom\big(\mathcal{O}_{\mathbb{P}^n}(-1)^a, \mathcal{O}_{\mathbb{P}^n}{}^b\big)$ *yielding a monad*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^a \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}{}^b \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^c \longrightarrow 0.$$

Furthermore, the set of pairs

$$(\alpha, \beta) \in Hom\big(\mathcal{O}_{\mathbb{P}^n}(-1)^a, \mathcal{O}_{\mathbb{P}^n}{}^b\big) \times Hom\big(\mathcal{O}_{\mathbb{P}^n}{}^b, \mathcal{O}_{\mathbb{P}^n}(1)^c\big)$$

yielding such a monad is an irreducible algebraic variety.

Simplicity of the cohomology sheaf

Given a monad $M_\bullet$, what can we say about its cohomology sheaf?

$$M_\bullet : \; 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

Simplicity of the cohomology sheaf

Theorem (Marchesi, Soares, __)

Let X be an ACM, smooth, n-dimensional, projective variety embedded in $\mathbb{P}^N$ by a very ample invertible sheaf L and let a and b be integers such that $\max\{n+1, a+1\} \leq b \leq h^0(L)$. Then there exists a monad

$$0 \longrightarrow (L^\vee)^a \xrightarrow{\alpha} {\mathcal{O}_X}^b \xrightarrow{\beta} L \longrightarrow 0.$$

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Moreover, when $b = n+1$, then any monad of this type is simple.

Cohomological characterisation of monads

Given a sheaf over X, does it come from a monad of type $M_\bullet$?

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Cohomological characterisation of monads

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$$^{\vee}\mathcal{B} = \big(\mathcal{O}_{\mathbb{P}^n}(d+n), \dots, \Lambda^i T_{\mathbb{P}^n}(d+n-i), \dots, \Lambda^n T_{\mathbb{P}^n}(d)\big).$$

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For $X = Q_n$, with $n \geq 3$,

$$\mathcal{B} = \big(\mathcal{O}_{Q_n}(-1), \Sigma_*(-1), \mathcal{O}_{Q_n}, \dots, \mathcal{O}_{Q_n}(n-2)\big),$$

$$^\vee\mathcal{B} = \big(\mathcal{O}_{Q_n}(n-2), \psi_1^*(n-2), \dots, \psi_{n-2}^*(n-2), \Sigma_*(n-1), \psi_0(n-1)\big)$$

where

$$\Sigma_*(a) = \begin{cases} \big(\Sigma_1(a), \Sigma_2(a)\big), & \text{if } n \text{ is even,} \\ \Sigma(a), & \text{if } n \text{ is odd.} \end{cases}$$

$$\psi_0 = \mathcal{O}_{Q_n}, \quad \psi_1 = \Omega^1(1)_{|Q_n}, \quad 0 \longrightarrow \Omega^j(j)_{|Q_n} \longrightarrow \psi_j \longrightarrow \psi_{j-2} \longrightarrow 0.$$

Cohomological characterisation of monads

Natural cohomology with respect to an *n*-block collection

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Natural cohomology with respect to an *n*-block collection

Hartshorne, Hirschowitz, 1982 We say that a coherent sheaf F over the projective space $\mathbb{P}^n$ has *natural cohomology* if for each $t \in \mathbb{Z}$, at most one of the cohomology groups $H^i\big(\mathbb{P}^n, F(t)\big)$ is non-zero.

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Definition

A coherent sheaf E on X has *natural cohomology with respect to* $\mathcal{B}$ if, for each $0 \leq p \leq n$ and $1 \leq l \leq \alpha_p$, there is at most one $q \geq 0$ such that $Ext^q\left(F_l^p, E\right)$ is non-zero.

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- $F_{i_0}^i$, $F_{j_0}^j$, $F_{k_0}^k$ elements of $\mathcal{B}$, of ranks $r_i, r_j, r_k \geq 1$, respectively;

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- $F_{i_0}^i$, $F_{j_0}^j$, $F_{k_0}^k$ elements of $\mathcal{B}$, of ranks $r_i, r_j, r_k \geq 1$, respectively;
- E torsion-free sheaf on X.

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Theorem (Soares, __ 2014)

Then E is the cohomology sheaf of a monad defined by

$$M_\bullet : \; 0 \longrightarrow \left(F_{i_0}^i\right)^a \longrightarrow \left(F_{j_0}^j\right)^b \longrightarrow \left(F_{k_0}^k\right)^c \longrightarrow 0,$$

if and only if

- $c_t(E) = c_t\left(F_{j_0}^j\right)^b c_t\left(F_{i_0}^i\right)^{-a} c_t\left(F_{k_0}^k\right)^{-c}$;
- $\text{rank}(E) = br_j - ar_i - cr_k$;
- *E has natural cohomology with respect to ${}^\vee\mathcal{B}$.*

Museu de Arte Contemporânea de Serralves, Álvaro Siza Vieira, 1997