Min-max methods in Geometry

André Neves

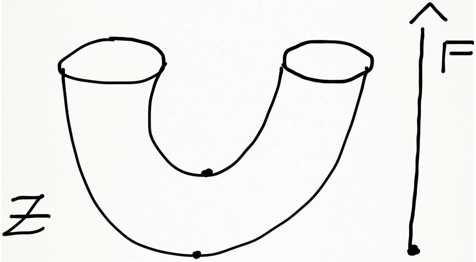
Imperial College London

Outline

1. Min-max theory overview
2. Applications in Geometry
3. Some new progress

Min-max Theory

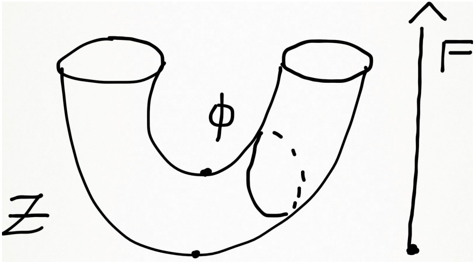
Consider a space $\mathcal{Z}$ and a functional $F : \mathcal{Z} \to [0, \infty]$. How to find critical points for F?



Key Idea (M. Morse): Topology of $\mathcal{Z}$ forces F to have certain critical points.

Min-max Theory

Consider a space $\mathcal{Z}$ and a functional $F : \mathcal{Z} \to [0, \infty]$. How to find critical points for F?

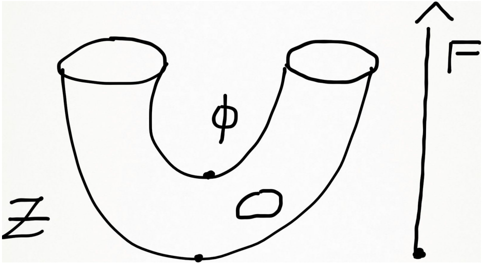


Key Idea (M. Morse): Topology of $\mathcal{Z}$ forces F to have certain critical points.

- X^k a topological space with dimension k and $\Phi : X^k \to \mathcal{Z}$ continuous;

Min-max Theory

Consider a space $\mathcal{Z}$ and a functional $F : \mathcal{Z} \to [0, \infty]$. How to find critical points for F?

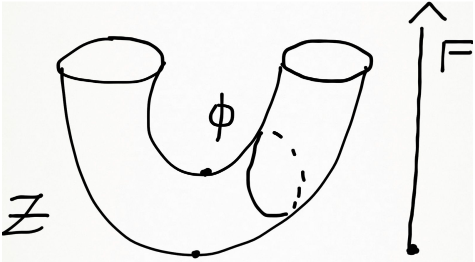


Key Idea (M. Morse): Topology of $\mathcal{Z}$ forces F to have certain critical points.

- X^k a topological space with dimension k and $\Phi : X^k \to \mathcal{Z}$ continuous;

Min-max Theory

Consider a space $\mathcal{Z}$ and a functional $F : \mathcal{Z} \to [0, \infty]$. How to find critical points for F?

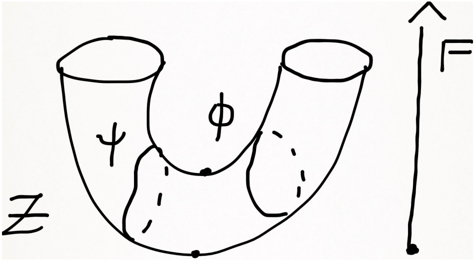


Key Idea (M. Morse): Topology of $\mathcal{Z}$ forces F to have certain critical points.

- X^k a topological space with dimension k and $\Phi : X^k \to \mathcal{Z}$ continuous;

Min-max Theory

Consider a space $\mathcal{Z}$ and a functional $F : \mathcal{Z} \to [0, \infty]$. How to find critical points for F?

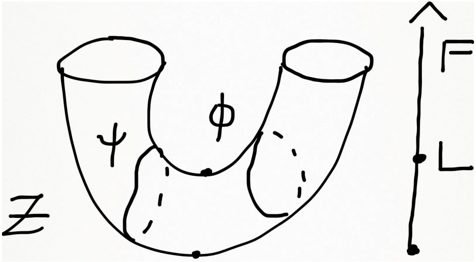


Key Idea (M. Morse): Topology of $\mathcal{Z}$ forces F to have certain critical points.

- X^k a topological space with dimension k and $\Phi : X^k \to \mathcal{Z}$ continuous;
- $[\Phi] = \{\text{all } \Psi \text{ homotopic to } \Phi \text{ relative to } \partial X^k\}$;

Min-max Theory

Consider a space $\mathcal{Z}$ and a functional $F : \mathcal{Z} \to [0, \infty]$. How to find critical points for F?

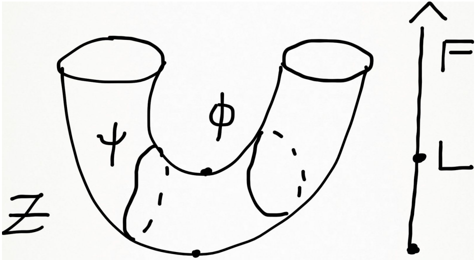


Key Idea (M. Morse): Topology of $\mathcal{Z}$ forces F to have certain critical points.

- X^k a topological space with dimension k and $\Phi : X^k \to \mathcal{Z}$ continuous;
- $[\Phi] = \{$all Ψ homotopic to Φ relative to $\partial X^k\}$;
- $L([\Phi]) = \inf_{\Psi \in [\Phi]} \max_{x \in X^k} F(\Psi(x))$;

Min-max Theory

Consider a space $\mathcal{Z}$ and a functional $F : \mathcal{Z} \to [0, \infty]$. How to find critical points for F?



Key Idea (M. Morse): Topology of $\mathcal{Z}$ forces F to have certain critical points.

- X^k a topological space with dimension k and $\Phi : X^k \to \mathcal{Z}$ continuous;
- $[\Phi] = \{$all Ψ homotopic to Φ relative to $\partial X^k\}$;
- $L([\Phi]) = \inf_{\Psi \in [\Phi]} \max_{x \in X^k} F(\Psi(x))$;
- $L([\Phi]) = F(z_0)$ for some critical point $z_0 \in \mathcal{Z}$ of Morse index at most k.

Min-max Theory

(M^n, g) closed compact Riemannian n-manifold, $3 \leq n \leq 7$.

The space

- $\mathcal{Z}_{n-1}(M) =$ "$\{$all compact oriented hypersurfaces in $M\}$";
- $\mathcal{Z}_{n-1}(M; \mathbb{Z}_2) =$ "$\{$all compact hypersurfaces in $M\}$";

Min-max Theory

(M^n, g) closed compact Riemannian n-manifold, $3 \leq n \leq 7$.

The space

- $\mathcal{Z}_{n-1}(M) =$ "{all compact oriented hypersurfaces in M}";
- $\mathcal{Z}_{n-1}(M; \mathbb{Z}_2) =$ "{all compact hypersurfaces in M}";

The function

- $F : \mathcal{Z}_{n-1}(M) \to [0, +\infty], \quad F(\Sigma) = vol(\Sigma)$;
- The critical points are called minimal hypersurfaces.

Min-max Theory

(M^n, g) closed compact Riemannian n-manifold, $3 \leq n \leq 7$.

The space

- $\mathcal{Z}_{n-1}(M) =$ "{all compact oriented hypersurfaces in M}";
- $\mathcal{Z}_{n-1}(M; \mathbb{Z}_2) =$ "{all compact hypersurfaces in M}";

The function

- $F : \mathcal{Z}_{n-1}(M) \to [0, +\infty], \quad F(\Sigma) = vol(\Sigma);$
- The critical points are called minimal hypersurfaces.

Almgren–Pitts Min-max Theorem

If the map $\Phi : X^k \to \mathcal{Z}_{n-1}(M)$ is nontrivial then there is an embedded minimal hypersurface Σ (with multiplicities) so that

$$\mathbf{L}([\Phi]) = vol(\Sigma).$$

Conjecture: Σ has index less or equal than k.

The sweepout

Theorem (Almgren, '62)

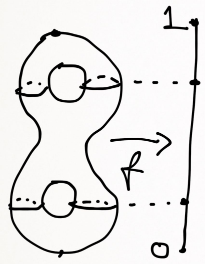
There is $\Phi_1 : S^1 \to \mathcal{Z}_{n-1}(M)$ *so that* $[\Phi_1]$ *is non-trivial.*

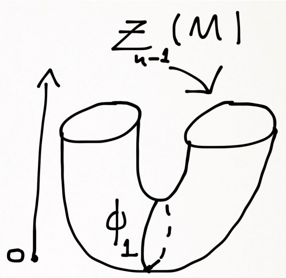
The sweepout

Theorem (Almgren, ’62)

There is $\Phi_1 : S^1 \to \mathcal{Z}_{n-1}(M)$ so that $[\Phi_1]$ is non-trivial.

- $f : M \to [0, 1]$ Morse function;
- $\Phi_1 : [0, 1] \to \mathcal{Z}_{n-1}(M)$, $\Phi_1(t) = f^{-1}(t)$.





The sweepout - Applications

(Poincaré, 1905) *Does every 2-sphere (S^2, g) admit a closed geodesic?*

- Foundational question in Geometry, Topology, and Dynamical Systems.

The sweepout - Applications

(Poincaré, 1905) *Does every* 2*-sphere* (S^2, g) *admit a closed geodesic?*

- Foundational question in Geometry, Topology, and Dynamical Systems.

Application 1 (Birkhoff, '17)

Every (S^2, g) admits a closed geodesic because $\Pi_1(\mathcal{Z}_1(S^2)) \neq 0$.

The sweepout - Applications

(Poincaré, 1905) *Does every 2-sphere* (S^2, g) *admit a closed geodesic?*

- Foundational question in Geometry, Topology, and Dynamical Systems.

Application 1 (Birkhoff, '17)

Every (S^2, g) admits a closed geodesic because $\Pi_1(\mathcal{Z}_1(S^2)) \neq 0$.

Application 2 (Pitts '81, Schoen-Simon '81)

Every (M^n, g) admits an embedded minimal hypersurface.

The multi-parameter sweepout

(Almgren, 60's) *$\mathcal{Z}_{n-1}(M; \mathbb{Z}_2)$ is weakly homotopic to $\mathbb{RP}^\infty$.*

Thus, for every $k \in \mathbb{N}$, there is $\Phi_k : \mathbb{RP}^k \to \mathcal{Z}_{n-1}(M; \mathbb{Z}_2)$ which is homotopically non-trivial.

The multi-parameter sweepout

(Almgren, 60's) *$\mathcal{Z}_{n-1}(M;\mathbb{Z}_2)$ is weakly homotopic to $\mathbb{RP}^\infty$.*

Thus, for every $k \in \mathbb{N}$, there is $\Phi_k : \mathbb{RP}^k \to \mathcal{Z}_{n-1}(M;\mathbb{Z}_2)$ which is homotopically non-trivial.

- The k-width of M is

$$\omega_k(M) := L([\Phi_k]) = \inf_{\{\Phi\in[\Phi_k]\}} \sup_{x\in\mathbb{RP}^k} vol(\Phi(x)), \quad k \in \mathbb{N}.$$

  Compare with k-th eigenvalue

$$\lambda_k(M) = \inf_{\{(k+1)\text{ plane } P\subset W^{1,2}\}} \sup_{f\in P-\{0\}} \frac{\int_M |\nabla f|^2}{\int_M f^2}.$$

The multi-parameter sweepout

(Almgren, 60's) *$\mathcal{Z}_{n-1}(M;\mathbb{Z}_2)$ is weakly homotopic to $\mathbb{RP}^\infty$.*

Thus, for every $k \in \mathbb{N}$, there is $\Phi_k : \mathbb{RP}^k \to \mathcal{Z}_{n-1}(M;\mathbb{Z}_2)$ which is homotopically non-trivial.

- The k-width of M is

$$\omega_k(M) := L([\Phi_k]) = \inf_{\{\Phi\in[\Phi_k]\}} \sup_{x\in\mathbb{RP}^k} vol(\Phi(x)), \quad k \in \mathbb{N}.$$

Compare with k-th eigenvalue

$$\lambda_k(M) = \inf_{\{(k+1)\text{ plane } P\subset W^{1,2}\}} \sup_{f\in P-\{0\}} \frac{\int_M |\nabla f|^2}{\int_M f^2}.$$

Theorem (Gromov, 70's, Guth, 07')

$\omega_k(M)$ grows like $k^{1/n}$ as k tends to infinity.

The multi-parameter sweepout - Applications

(Franks, '92, Bangert, '93) *Every* (S^2, g) *admits an infinite number of closed geodesics*.

The multi-parameter sweepout - Applications

(Franks, '92, Bangert, '93) *Every* (S^2, g) *admits an infinite number of closed geodesics*.

(Yau Conjecture, '82) *Every compact* 3*-manifold* (M, g) *admits an infinite number of distinct minimal surfaces.*

- (Khan–Markovic, '11) *Compact hyperbolic* 3*-manifolds have an infinite number of minimal immersed surfaces.*

The multi-parameter sweepout - Applications

(Franks, '92, Bangert, '93) *Every* (S^2, g) *admits an infinite number of closed geodesics*.

(Yau Conjecture, '82) *Every compact* 3*-manifold* (M, g) *admits an infinite number of distinct minimal surfaces.*

- (Khan–Markovic, '11) *Compact hyperbolic* 3*-manifolds have an infinite number of minimal immersed surfaces.*

Application (Marques–N., 2013)

Assume (M^n, g) *compact manifold with positive Ricci curvature.*

Then M *admits an infinite number of distinct embedded minimal hypersurfaces.*

- These minimal hypersurfaces are obtained by finding minimal hypersurface Σ_k such that $\omega_k(M) = \mathrm{vol}(\Sigma_k)$.

  The key issue is to show that they are geometrically distinct because $k\Sigma$ is also a critical point for volume functional for all $k \in \mathbb{N}$.

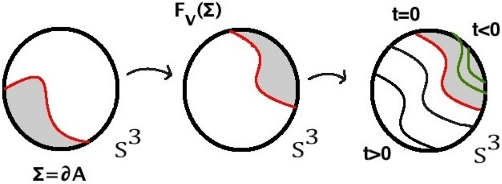
Canonical family for surfaces in $\mathcal{Z}_2(S^3)$

- Every $v \in B^4$ gives a conformal dilation F_v of S^3 centered at $\{\frac{v}{|v|}, -\frac{v}{|v|}\}$.

Canonical family for surfaces in $\mathcal{Z}_2(S^3)$

- Every $v \in B^4$ gives a conformal dilation F_v of S^3 centered at $\{\frac{v}{|v|}, -\frac{v}{|v|}\}$.
- Given Σ embedded surface in S^3 consider

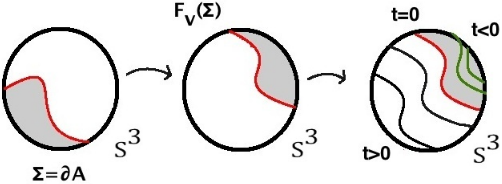
$$\mathcal{C} : B^4 \times [-\pi, \pi] \to \mathcal{Z}_2(S^3), \quad \mathcal{C}(v, t) = \text{“surface” at distance } t \text{ from } F_v(\Sigma).$$



Canonical family for surfaces in $\mathcal{Z}_2(S^3)$

- Every $v \in B^4$ gives a conformal dilation F_v of S^3 centered at $\{\frac{v}{|v|}, -\frac{v}{|v|}\}$.
- Given Σ embedded surface in S^3 consider

$$\mathcal{C} : B^4 \times [-\pi, \pi] \to \mathcal{Z}_2(S^3), \quad \mathcal{C}(v, t) = \text{“surface” at distance } t \text{ from } F_v(\Sigma).$$



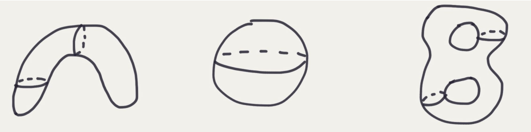
- $\mathcal{C}$ maps the boundary of $B^4 \times [-\pi, \pi]$ into $\mathcal{R} = \{\text{round spheres in } S^3\}$. Thus $[\mathcal{C}]$ lies in $\Pi_5(\mathcal{Z}_2(S^3), \mathcal{R})$.

Theorem (Marques-N., ’12)

If Σ has **positive genus** *then $\mathcal{C}$ cannot be deformed into the space of all round spheres, i.e., $[\mathcal{C}] \neq 0$ in $\Pi_5(\mathcal{Z}_2(S^3), \mathcal{R})$.*

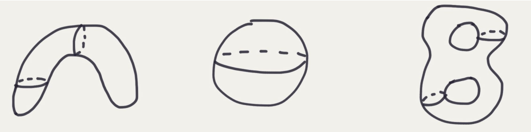
Canonical family for surfaces - Applications

What is the best immersion of a surface in space?



Canonical family for surfaces - Applications

What is the best immersion of a surface in space?



Pick the one that minimizes the bending energy (known as Willmore energy)

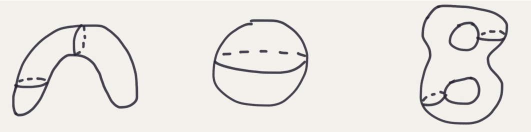
$$W(\Sigma) := \int_\Sigma \left(\frac{k_1 + k_2}{2} \right)^2 dx,$$

where k_1, k_2 are the principal curvatures of Σ

- The bending energy is invariant under conformal transformations, a larger group than the set of isometries.

Canonical family for surfaces - Applications

What is the best immersion of a surface in space?



Pick the one that minimizes the bending energy (known as Willmore energy)

$$W(\Sigma) := \int_{\Sigma} \left(\frac{k_1 + k_2}{2} \right)^2 dx,$$

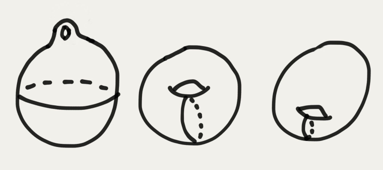
where k_1, k_2 are the principal curvatures of Σ

- The bending energy is invariant under conformal transformations, a larger group than the set of isometries.

(Willmore, '65) *Every compact surface Σ has $W(\Sigma) \geq 4\pi$ with equality only for round spheres.*

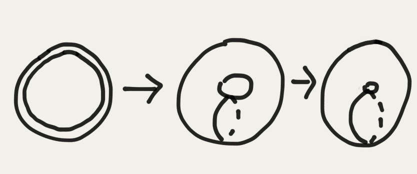
Canonical family for surfaces - Applications

(Willmore, ’65) *What is the least bended torus in space?*



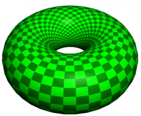
Canonical family for surfaces - Applications

(Willmore, '65) *What is the least bended torus in space?*



Canonical family for surfaces - Applications

(Willmore, '65) *What is the least bended torus in space?*

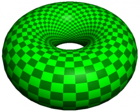


Willmore Conjecture, '65

Every torus Σ *in space has* $W(\Sigma) \geq 2\pi^2$ *with equality only for conformal images of that optimal torus.*

Canonical family for surfaces - Applications

(Willmore, '65) *What is the least bended torus in space?*



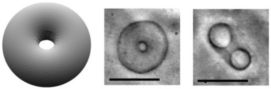
Willmore Conjecture, '65

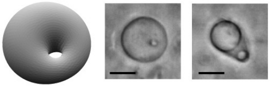
Every torus Σ in space has $W(\Sigma) \geq 2\pi^2$ with equality only for conformal images of that optimal torus.

Application 1 (Marques-N., '12)

The Willmore conjecture holds.

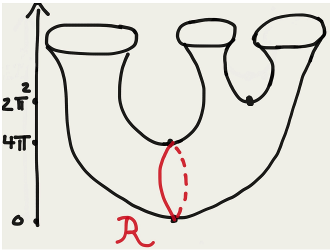
Observed in Biophysics by Bensimon and Mutz in 1992.





Willmore Conjecture and Min-max theory

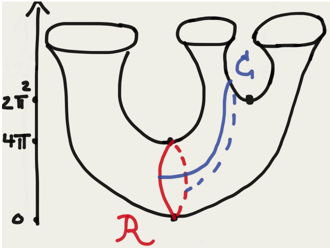
(Urbano, '90) *Minimal surfaces in* S^3 *with Morse index* ≤ 5 *are either equators (index* 1 *and area* 4π*) or Clifford tori (index* 5 *and area* $2\pi^2$*).*



- $\mathcal{R} = \{$round spheres in $S^3\}$;

Willmore Conjecture and Min-max theory

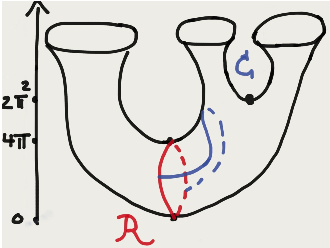
(Urbano, '90) *Minimal surfaces in S^3 with Morse index ≤ 5 are either equators (index 1 and area 4π) or Clifford tori (index 5 and area $2\pi^2$).*



- $\mathcal{R} = \{$round spheres in $S^3\}$;
- Given a torus Σ consider the canonical family $\mathcal{C}$.

Willmore Conjecture and Min-max theory

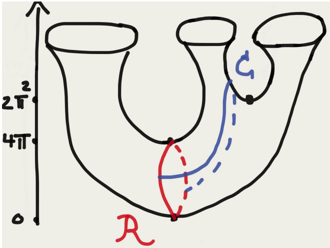
(Urbano, '90) *Minimal surfaces in S^3 with Morse index ≤ 5 are either equators (index 1 and area 4π) or Clifford tori (index 5 and area $2\pi^2$).*



- $\mathcal{R} = \{$round spheres in $S^3\}$;
- Given a torus Σ consider the canonical family $\mathcal{C}$.

Willmore Conjecture and Min-max theory

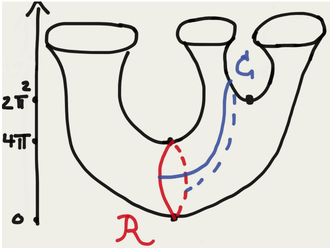
(Urbano, '90) *Minimal surfaces in S^3 with Morse index ≤ 5 are either equators (index 1 and area 4π) or Clifford tori (index 5 and area $2\pi^2$).*



- $\mathcal{R} = \{$round spheres in $S^3\}$;
- Given a torus Σ consider the canonical family $\mathcal{C}$.

Willmore Conjecture and Min-max theory

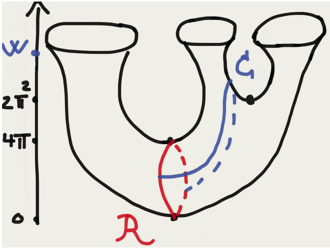
(Urbano, '90) *Minimal surfaces in S^3 with Morse index ≤ 5 are either equators (index 1 and area 4π) or Clifford tori (index 5 and area $2\pi^2$).*



- $\mathcal{R} = \{$round spheres in $S^3\}$;
- Given a torus Σ consider the canonical family $\mathcal{C}$. Because $\mathcal{C}$ cannot be homotoped into $\mathcal{R}$ we have $2\pi^2 \leq \sup_x \mathit{area}(\mathcal{C}(x))$

Willmore Conjecture and Min-max theory

(Urbano, '90) *Minimal surfaces in S^3 with Morse index ≤ 5 are either equators (index 1 and area 4π) or Clifford tori (index 5 and area $2\pi^2$).*



- $\mathcal{R} = \{$round spheres in $S^3\}$;
- Given a torus Σ consider the canonical family $\mathcal{C}$. Because $\mathcal{C}$ cannot be homotoped into $\mathcal{R}$ we have $2\pi^2 \leq \sup_x area(\mathcal{C}(x))$
- $\mathcal{C}$ is such that $\sup_x area(\mathcal{C}(x)) \leq W(\Sigma) =$Willmore energy of Σ and thus

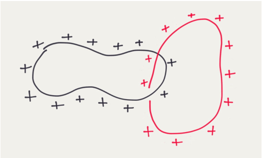
$$2\pi^2 \leq \sup_x area(\mathcal{C}(x)) \leq W(\Sigma)$$

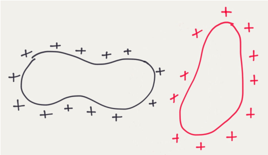
Canonical family for surfaces - Applications

“*The behavior of a charged loop of strings is a traditional subject for tea-time speculation.*” (Freedman–He–Wang, ’94)

Canonical family for surfaces - Applications

"*The behavior of a charged loop of strings is a traditional subject for tea-time speculation.*" (Freedman–He–Wang, '94)

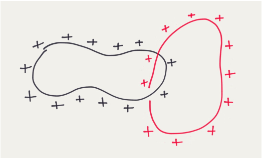


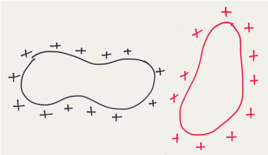


To every link in space (γ_1, γ_2), one can associate a Möbius cross energy $E(\gamma_1, \gamma_2)$ that is also conformally invariant.

Canonical family for surfaces - Applications

"*The behavior of a charged loop of strings is a traditional subject for tea-time speculation.*" (Freedman–He–Wang, '94)





To every link in space (γ_1, γ_2), one can associate a Möbius cross energy $E(\gamma_1, \gamma_2)$ that is also conformally invariant.

Freedman–He–Wang Conjecture, '94

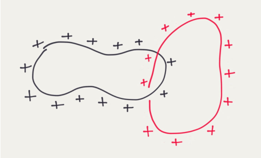
Every non-trivial linked curves γ_1, γ_2 *have* $E(\gamma_1, \gamma_2) \geq 2\pi^2$ *with equality achieved only by conformal transformations of the Hopf-link.*

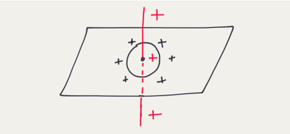
Application 2 (Agol–Marques–N., '12)

The conjecture holds.

Canonical family for surfaces - Applications

"*The behavior of a charged loop of strings is a traditional subject for tea-time speculation.*" (Freedman–He–Wang, '94)





To every link in space (γ_1, γ_2), one can associate a Möbius cross energy $E(\gamma_1, \gamma_2)$ that is also conformally invariant.

Freedman–He–Wang Conjecture, '94

Every non-trivial linked curves γ_1, γ_2 have $E(\gamma_1, \gamma_2) \geq 2\pi^2$ with equality achieved only by conformal transformations of the Hopf-link.

Application 2 (Agol–Marques–N., '12)

The conjecture holds.

New Progress

Unfortunately, Almgren-Pitts Theory does not provide index estimates.

New Progress

Unfortunately, Almgren-Pitts Theory does not provide index estimates.

Theorem A (Marques-N.)

Assume that every embedded hypersurfaces in M is orientable.

Let $\Phi : S^1 \to \mathcal{Z}_{n-1}(M)$ be a sweepout. For a generic set of metrics, there is minimal embedded hypersurface Σ such that

$$\mathbf{L}([\Phi]) = vol(\Sigma) \quad \text{and } index(\Sigma) = 1.$$

New Progress

Unfortunately, Almgren-Pitts Theory does not provide index estimates.

Theorem A (Marques-N.)

Assume that every embedded hypersurfaces in M is orientable.

Let $\Phi : S^1 \to \mathcal{Z}_{n-1}(M)$ be a sweepout. For a generic set of metrics, there is minimal embedded hypersurface Σ such that

$$\mathbf{L}([\Phi]) = vol(\Sigma) \quad \textit{and index}(\Sigma) = 1.$$

Theorem B (Marques-N.)

Let $\Phi : X^k \to \mathcal{Z}_{n-1}(M)$ be nontrivial. There is a minimal embedded hypersurface Σ such that

$$\mathbf{L}([\Phi]) = vol(\Sigma) \quad \textit{and index}(\textit{support}\,\Sigma) \leq k.$$

Some Questions

- If you classify low index minimal surfaces, then Min-max theory should give you global consequences.

Some Questions

- If you classify low index minimal surfaces, then Min-max theory should give you global consequences.
- For instance, if you classify
  1. minimal surfaces in S^3 with index ≤ 9;
  2. minimal surfaces in S^4 with index ≤ 6;

  then you should prove
  1. Willmore conjecture for genus 2 surfaces;
  2. Willmore conjecture in $\mathbb{R}^4$;

Some Questions

- If you classify low index minimal surfaces, then Min-max theory should give you global consequences.
- For instance, if you classify
  1. minimal surfaces in S^3 with index ≤ 9;
  2. minimal surfaces in S^4 with index ≤ 6;

  then you should prove
  1. Willmore conjecture for genus 2 surfaces;
  2. Willmore conjecture in $\mathbb{R}^4$;

Some Questions

- If you classify low index minimal surfaces, then Min-max theory should give you global consequences.
- For instance, if you classify
  1. minimal surfaces in S^3 with index ≤ 9;
  2. minimal surfaces in S^4 with index ≤ 6;

  then you should prove
  1. Willmore conjecture for genus 2 surfaces;
  2. Willmore conjecture in $\mathbb{R}^4$;
- Does every (S^4, g) admits a minimal embedded hypersphere? Probably too hard... What about a minimal hypersurface Σ with $b_1(\Sigma) \leq 1$?

Some Questions

- If you classify low index minimal surfaces, then Min-max theory should give you global consequences.
- For instance, if you classify
  1. minimal surfaces in S^3 with index ≤ 9;
  2. minimal surfaces in S^4 with index ≤ 6;

  then you should prove
  1. Willmore conjecture for genus 2 surfaces;
  2. Willmore conjecture in $\mathbb{R}^4$;
- Does every (S^4, g) admits a minimal embedded hypersphere? Probably too hard... What about a minimal hypersurface Σ with $b_1(\Sigma) \leq 1$?
- Does every (S^4, g) admits an infinite number of minimal branched 2-spheres?

Some Questions

- Does every (S^4, g) admits a minimal embedded hypersphere? Probably too hard... What about a minimal hypersurface Σ with $b_1(\Sigma) \leq 1$?
- Does every (S^4, g) admits an infinite number of minimal branched 2-spheres?

Some Questions

- Does every (S^4, g) admits a minimal embedded hypersphere? Probably too hard... What about a minimal hypersurface Σ with $b_1(\Sigma) \leq 1$?
- Does every (S^4, g) admits an infinite number of minimal branched 2-spheres?

New Questions

- Is there any relation between the spectrum of (M, g) and the sequence of k-widths $\omega_k(M)$?

New Questions

- Is there any relation between the spectrum of (M, g) and the sequence of k-widths $\omega_k(M)$?

- If $\phi_0, \ldots, \phi_k$ are the first k eigenfunctions then

$$\omega_k(M) \leq \sup_{a_0,\ldots,a_k \in \mathbb{R}} vol\{a_0\phi_0 + \ldots + a_k\phi_k = 0\}.$$

Is the right side asymptotically sharp? If true would answer an old question of Yau regarding the size of nodal sets.

New Questions

- Is there any relation between the spectrum of (M, g) and the sequence of k-widths $\omega_k(M)$?

- If $\phi_0, \ldots, \phi_k$ are the first k eigenfunctions then

$$\omega_k(M) \leq \sup_{a_0,\ldots,a_k \in \mathbb{R}} vol\{a_0\phi_0 + \ldots + a_k\phi_k = 0\}.$$

  Is the right side asymptotically sharp? If true would answer an old question of Yau regarding the size of nodal sets.

- Does $\{\omega_k(M)\}_{k\in\mathbb{N}}$ satisfy a Weyl Law?

$$\lim_{k\to\infty} \frac{\omega_k(M)}{k^{\frac{1}{n}}} = \alpha(n)(vol\, M)^{\frac{n-1}{n}}.$$

New Questions

- Is there any relation between the spectrum of (M, g) and the sequence of k-widths $\omega_k(M)$?

- If $\phi_0, \ldots, \phi_k$ are the first k eigenfunctions then
$$\omega_k(M) \leq \sup_{a_0, \ldots, a_k \in \mathbb{R}} vol\{a_0\phi_0 + \ldots + a_k\phi_k = 0\}.$$
  Is the right side asymptotically sharp? If true would answer an old question of Yau regarding the size of nodal sets.

- Does $\{\omega_k(M)\}_{k\in\mathbb{N}}$ satisfy a Weyl Law?
$$\lim_{k\to\infty} \frac{\omega_k(M)}{k^{\frac{1}{n}}} = \alpha(n)(vol\, M)^{\frac{n-1}{n}}.$$

- We expect that for generic metrics $\omega_k(M)$ is attained by a multiplicity one minimal hypersurface Σ_k with index k. What can we say about Σ_k?

New Questions

- Is there any relation between the spectrum of (M, g) and the sequence of k-widths $\omega_k(M)$?

- If $\phi_0, \ldots, \phi_k$ are the first k eigenfunctions then

$$\omega_k(M) \leq \sup_{a_0, \ldots, a_k \in \mathbb{R}} vol\{a_0\phi_0 + \ldots + a_k\phi_k = 0\}.$$

  Is the right side asymptotically sharp? If true would answer an old question of Yau regarding the size of nodal sets.

- Does $\{\omega_k(M)\}_{k\in\mathbb{N}}$ satisfy a Weyl Law?

$$\lim_{k\to\infty} \frac{\omega_k(M)}{k^{\frac{1}{n}}} = \alpha(n)(vol\, M)^{\frac{n-1}{n}}.$$

- We expect that for generic metrics $\omega_k(M)$ is attained by a multiplicity one minimal hypersurface Σ_k with index k. What can we say about Σ_k?

  We expect Σ_k to become evenly distributed and to have first Betti number proportional to k.