Recent developments on certain dispersive equations as infinite dimensional Hamiltonian systems

Gigliola Staffilani

Massachusetts Institute of Technology

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Introduction

In this talk I will use the periodic semi linear Schrödinger Cauchy problem

$$\begin{cases} iu_t + \Delta u = \lambda |u|^{p-1} u, \\ u(x,0) = u_0(x), \quad x \in \mathbb{T}^n, \end{cases} \tag{1.1}$$

where $u_0(x)$ is the initial profile, $\lambda = \pm 1$, $p > 1$, $u : \mathbb{R} \times \mathbb{T}^n \to \mathbb{C}$, and $\mathbb{T}^n$ is a n-dimensional torus[1], to illustrate how a partial differential equation that may have been introduced to model a certain phenomenon in physics may also have structures that touch many different areas of mathematics like:

Fourier and harmonic analysis, analytic number theory, probability, dynamical systems and symplectic geometry.

For obvious reasons here I will only present the *simple* connections of problem (1.1) with the areas of mathematics listed above and one should not think that this is all there is. On the contrary all of these connections are very active areas of research at the moment with many open problems.

[1] Later we will distinguish between a rational and an irrational torus.

Bose Einstein Condensate

The Cauchy problem

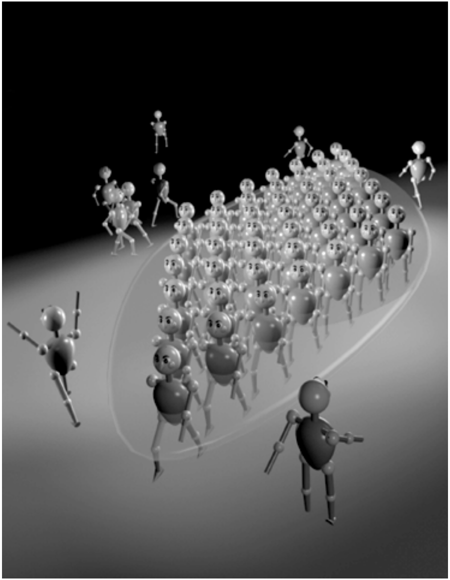
$$
\begin{cases} iu_t + \Delta u = \lambda |u|^2 u, \\ u(x,0) = u_0(x), \quad x \in \mathbb{T}^3 \end{cases} \tag{1.2}
$$

is used to describe several phenomena, but in particular is the problem that *governs* the Bose Einstein Condensate (BEC).

The BEC is the state of matter of a dilute gas of weakly interacting bosons confined in an external potential and cooled to temperatures very near absolute zero. The pointwise density of this *gas* at time t is represented by $|u(x,t)|^2$.

Physically what happens is that the bosons particles, that are in a chaotic state when the temperature T is above absolute zero, as $T \to 0$ start rearranging themselves, losing their *"individual identity"* and shaping themselves as a wave $u(x,t)$ that solves an equation as in (1.2).

Rendering of Bose Einstein Condensate



Science (December 22, 1995) by Steve Keller.

The Basic Questions

When a Cauchy problem is proposed as a model the obvious question to study is *well-posedness*, that is

- Existence of solutions
- Uniqueness of solutions
- Stability of solutions

One way to address the question of well-posedness is by rewriting the Cauchy problem (1.1) as the integral equation given by the Duhamel principle

$$u(x,t) = S(t)u_0(x) + c\int_0^t S(t-t')|u|^{p-1}u(x,t'))\,dt', \tag{1.3}$$

where $S(t)u_0(x)$ is the solution of the associated linear problem that we will discuss in details below.

The idea then is to use a fixed point theorem in a space of functions whose norm is dictated by strong estimates for $S(t)u_0(x)$.

The linear solution $S(t)u_0(x)$ in $\mathbb{T}$

We first recall that $S(t)u_0(x) = v(x,t)$, where $v(x,t)$ solves the Cauchy problem

(1.4)
$$\begin{cases} iv_t + \Delta v = 0, \\ v(x,0) = u_0(x), \end{cases}$$

for simplicity we assume that $x \in \mathbb{T}$. Using Fourier series we write the solution

$$v(x,t) = \sum_{k \in \mathbb{Z}} a(k,t)e^{ikx}, \quad \text{and} \quad a(k,t) = \hat{v}(k,t).$$

If we take the Fourier transform of (1.4), for every frequency k we obtain an ODE for $a(k,t)$:

(1.5)
$$\begin{cases} i\frac{d}{dt}a(k,t) + (-ik)^2 a(k,t) = 0, \\ a(k,0) = \hat{u}_0(k), \end{cases}$$

and the solution becomes

$$a(k,t) = \hat{u}_0(k)e^{itk^2} \quad \text{and} \quad v(x,t) = \sum_{k \in \mathbb{Z}} \hat{u}_0(k)e^{i(kx+tk^2)}.$$

The linear solution $S(t)u_0$ in $\mathbb{T}^n$

Now we assume that $x \in \mathbb{T}^n$ and that $c_i > 0,\ i = 1, ..., n$ are the periods. Then if we repeat the same argument we obtain that

$$S(t)u_0(x,t) = \sum_{k \in \mathbb{Z}} \hat{u}_0(k) e^{i(kx + t\gamma(k))},$$

where

$$\gamma(k) = \sum_{i=1}^{n} c_i k_i^2.$$

It will be relevant for later to observe that in the special case when $c_i = 1,\ i = 1, ..., n$

$$\gamma(k) = R^2$$

represents the sphere of radius R.
If $c_i \in \mathbb{N},\ i = 1, ..., n$ we will call the torus $\mathbb{T}^n$ a *rational* torus, otherwise we will call it *irrational*.

Strichartz Estimates

The Strichartz estimates in $\mathbb{T}^n$ are non trivial estimates for $S(t)u_0(x)$. They where originally introduced by **Bourgain** as a conjecture that he also partially proved.

Conjecture

Assume that $\mathbb{T}^n$ is a rational or rational torus and the support of $\hat{u}_N$ is a ball $\{|n - n_0| \lesssim N\}$. Then

$$\|S(t)u_N\|_{L^q_t L^q_x([0,1]\times\mathbb{T}^n)} \lesssim C_q \|u_N\|_{L^2_x(\mathbb{T}^n)} \quad \text{if} \quad q < \frac{2(n+2)}{n}$$

$$\|S(t)u_N\|_{L^q_t L^q_x([0,1]\times\mathbb{T}^n)} \ll N^{\epsilon} \|u_N\|_{L^2_x(\mathbb{T}^n)} \quad \text{if} \quad q = \frac{2(n+2)}{n}$$

$$\|S(t)u_N\|_{L^q_t L^q_x([0,1]\times\mathbb{T}^n)} \lesssim C_q N^{\frac{n}{2} - \frac{n+2}{q}} \|u_N\|_{L^2_x(\mathbb{T}^n)} \quad \text{if} \quad q > \frac{2(n+2)}{n}$$

Remark

By some sort of stability intuition one expects that there shouldn't be any difference between a rational or irrational torus in proving the conjecture.

An Example: the L^4 Strichartz Estimates in $\mathbb{T}^2$

Let's concentrate on the L^4 Strichartz estimate that Bourgain proved in the 90's for a *rational* torus $\mathbb{T}^2$:

$$\|S(t)u_0\|_{L^4_{\mathbb{T}}L^4_T} \leq \|u_0\|_{H^\epsilon_x}.$$

We are not going to repeat the proof, here we only say that it is based on counting $\mathbb{Z}^2$ lattice points on ellipses given by

$$\gamma(k) = c_1 k_1^2 + c_2 k_2^2 = R^2$$

for $R \gg 1$. It is here that the rationality of the torus comes into play. In fact if the torus is rational, that is $c_j \in \mathbb{N}$, one can use some standard results from analytic number theory and obtain the sharp bound

$$\#\{k \in \mathbb{Z}^2 / \gamma(k) = R^2\} \sim expC \frac{\log R}{\log\log R}.$$

For irrational tori, that is when we simply assume $c_j \in \mathbb{R}^+$, only partial results were known till recently. See **Bourgain, Catoire-Wang, Z. Guo-Oh-Y. Wang** and **Demirbas** for the periodic case and **Burq-Gerard-Tzvetkov, Hani** for compact manifolds.

Proof of the Conjecture up to an ϵ Loss

Theorem (Bourgain-Demeter, 2014)

Assume that $\mathbb{T}^n$ is any torus of dimension n, rational or rational and assume that the support of $\hat{u}_N$ is in a ball $\{|n - n_0| \lesssim N\}$. Then

$$\|S(t)u_N\|_{L^q_t L^q_x([0,1]\times\mathbb{T}^n)} \lesssim C_q N^{\epsilon}\|u_N\|_{L^2_x(\mathbb{T}^n)} \quad \text{if} \quad q < \frac{2(n+2)}{n}$$

$$\|S(t)u_N\|_{L^q_t L^q_x([0,1]\times\mathbb{T}^n)} \ll N^{\epsilon}\|u_N\|_{L^2_x(\mathbb{T}^n)} \quad \text{if} \quad q = \frac{2(n+2)}{n}$$

$$\|S(t)u_N\|_{L^q_t L^q_x([0,1]\times\mathbb{T}^n)} \lesssim C_q N^{\frac{n}{2}-\frac{n+2}{q}} N^{\epsilon}\|u_N\|_{L^2_x(\mathbb{T}^n)} \quad \text{if} \quad q > \frac{2(n+2)}{n}$$

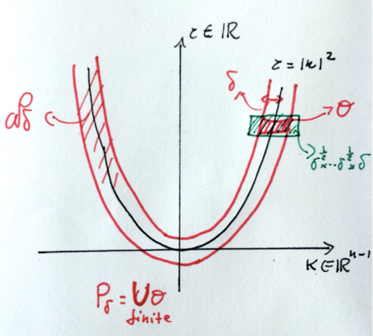
Remark

The proof of this theorem relies on some sharp estimates from harmonic analysis and incidence geometry theory, it does not use analytic number theory. But recently analytic number theory has been used by **Killip** *and* **Visan** *to recover the ϵ loss of derivative.*

See also related work by **Wolff**, **Bourgain-Guth** and **Garriós-Seeger**.

The Estimate from Harmonic Analysis

Let S be a compact C^2 hyper surface in $\mathbb{R}^n$ with positive definite second fundamental form. Let $0 < \delta \ll 1$ and $\mathcal{N}_\delta$ a δ-neighborhood of S and $\mathcal{P}_\delta$ a finitely overlapping cover of $\mathcal{N}_\delta$ with certain curved regions θ, (each θ sits inside a rectangular box of size $\sim \delta^{\frac{1}{2}} \times \cdots \times \delta$).



Decoupling Estimate

We can now state the theorem:

Theorem (Decoupling Estimate)

If supp$(\hat{f}) \subset \mathcal{N}_\delta$ *then for* $p \geq \frac{2(n+1)}{n-1}$ *and* $\epsilon \ll 1$

$$\|f\|_p \lesssim_\epsilon \delta^{-\frac{n-1}{2}+\frac{n+1}{2p}-\epsilon} \left(\sum_{\theta \in \mathcal{P}_\delta} \|f_\theta\|_p^2 \right)^{\frac{1}{2}},$$

where f_θ *is the Fourier restriction of f on the region* θ.

Remark

The fact that Strichartz estimates for irrational tori are not proved in a more direct way does not allow enough flexibility to treat other kids of questions, besides well-posedness. More on this later.

Local and global solutions

The norm L^2 that appear on the right hand side of the Strichartz estimates is not there by chance. In fact for the Schrödingier equation

$$iu_t + \Delta u = \lambda |u|^{p-1} u$$

the integral

$$M(u(t)) = \int |u(x,t)|^2 \, dx$$

is the *Mass* and it is conserved. As a consequence the most natural space for the initial data u_0 is the L^2 space. Unfortunately though at this level of regularity often little can be done to control nonlinear interactions and obtain well-posedness.
The next most relevant space is the Sobolev H^1 space. In fact the equation above keeps also the Energy (Hamiltonian) conserved:

$$H(u(t)) = \frac{1}{2} \int |\nabla u|^2(x,t) \, dx + \frac{2\lambda}{p+1} \int |u(t,x)|^{p+1} \, dx$$

Focusing and Defocusing

$$H(u(t)) = \frac{1}{2}\int |\nabla u|^2(x,t)\,dx + \frac{2\lambda}{p+1}\int |u(t,x)|^{p+1}\,dx$$

- If $\lambda = -1$ (Focusing) the energy could be negative and blow up may occur.
- If $\lambda = 1$ (Defocusing) the energy and the mass give a global in time bound for the the H^1 norm of $u(x,t)$. This bound and Strichartz estimates are then used to prove theorems such as:

Theorem (**Bourgain)**

The Cauchy problem

$$\begin{cases} iu_t + \Delta u = |u|^2 u, & \\ u(x,0) = u_0(x), & x \in \mathbb{T}^2 \quad \textit{rational and irrational} \end{cases}$$

is globally well-posed for data u_0 in H^1.

Now that we know that this equation has a global flow $u_0 \to u(x,t)$ we can start asking questions on the behavior of $u(x,t)$ in time.

Notion of Weak Turbulence

Definition

For this talk *Weak Turbulence* is the phenomenon of global-in-time solutions shifting their support towards increasingly high frequencies.

This shift is also called forward cascade.

- To measure weak turbulence we consider the function

$$\|u(t)\|_{\dot{H}^s}^2 = \int |\hat{u}(t,k)|^2 |k|^{2s} dk$$

  for $s \gg 1$ and prove that it grows for large times t.
- Weak turbulence is incompatible with scattering or complete integrability.

There are two theorems that summarize what we know for this particular problem. The first gives some polynomial in time bounds for $\|u(t)\|_{\dot{H}^s}^2$, the second[2] shows some kind of growth for certain solutions to the Cauchy problem above.

[2] See also recent results in this context by **Gerard-Grellier** and **Pocovnicu.**

Two Theorems on Weak Turbulence

Theorem (**Bourgain**, **Sohinger**)

Let u be the global solution of the cubic, defocusing, NLS equation on $\mathbb{T}^2$:

$$(3.1) \qquad \begin{cases} (i\partial_t + \Delta)u = |u|^2 u \\ u(0,x) = u_0(x), \quad \text{where } x \in \mathbb{T}^2, \end{cases}$$

Then

$$\|u(t)\|_{H^s(\mathbb{T}^2)} \lesssim (1+|t|)^{s+} \|u_0\|_{H^s(\mathbb{T}^2)}.$$

Theorem (**Colliander-Keel-S.-Takaoka-Tao**)

Assume $\mathbb{T}^2$ is **rational**. *Let $s > 1$, $K \gg 1$ and $0 < \sigma < 1$ be given. Then there exist a global smooth solution $u(x,t)$ to the defocusing IVP (3.1) above and $T > 0$ such that $\|u_0\|_{H^s} \leq \sigma$ and $\|u(T)\|_{H^s}^2 \geq K$.*

Related to the second theorem see also a recent work of **Guardia-Kaloshin** and **Haus-Procesi**.

The toy model

Here we only give few ideas that are relevant for the proof of the second theorem.
We start by making the ansatz

$$v(t,x) = \sum_{k\in\mathbb{Z}^2} a_k(t) e^{i(\langle k,x\rangle + |k|^2 t)},$$

and by rewriting the equation as an ODE in terms of the infinite vector $(a_k(t))$.
We also consider only the resonant part of the ODE and we construct a special finite set of frequencies Λ that is closed under resonant interactions and has several other "good" properties. Thanks to these properties we arrive to a finite dimensional toy model

$$-i\partial_t b_j(t) = -b_j(t)|b_j(t)|^2 - 2b_{j-1}(t)^2\overline{b_j(t)} - 2b_{j+1}(t)^2\overline{b_j(t)},$$

for $j = 0, ..., M+1$, with the boundary condition

$$b_0(t) = b_{M+1}(t) = 0.$$

Remark

This new system conserves the momentum, the mass ($\sum_{j=1}^{M} |b_j(t)|^2 = 1$) and the energy!

Global well-posedness for this system is not an issue. Then we define

$$\Sigma = \{x \in \mathbb{C}^M \ / \ |x|^2 = 1\} \text{ and } \ W(t) : \Sigma \to \Sigma,$$

where $W(t)b(0) = b(t)$ for any solution $b(t)$ of our system. It is easy to see that if we define the torus

$$\mathbb{T}_j = \{(b_1, ..., b_M) \in \Sigma \, / \, |b_j| = 1, \ b_k = 0, \ k \neq j\}$$

then

$$W(t)\mathbb{T}_j = \mathbb{T}_j \ \text{ for all } j = 1, ..., M$$

($\mathbb{T}_j$ is invariant).

At this point the problem has been set up in such a way that if we could show that once we start “near” one of the first tori (low frequencies) we end up at a certain time T near one of the last tori (high frequencies) then we are done. In fact we have the following result:

Theorem (Sliding theorem)

Let $M \geq 6$. Given $\epsilon > 0$ there exist x_3 within ϵ of $\mathbb{T}_3$ and x_{M-2} within ϵ of $\mathbb{T}_{M-2}$ and a time T such that

$$W(T)x_3 = x_{M-2}.$$

What the theorem says is that $W(t)x_3$ is a solution of total mass 1 arbitrarily concentrated near mode $j = 3$ at some time 0 that gets moved so that it is concentrated near mode $j = N - 2$ at later time T.

Remark

Recently **Hani-Pausader-Tzvetkov-Visciglia** *were able to use the actual solution of the Toy Model problem above to show a precise growth in time of the solution to an NLS equation in a mixed manifold $\mathbb{R} \times \mathbb{T}^2$.*

Finite Dimensional Hamiltonian Systems

Hamilton's equations of motion have the antisymmetric form

$$\dot{q}_i = \frac{\partial H(p,q)}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H(p,q)}{\partial q_i}$$

the Hamiltonian $H(p,q)$ being a first integral:

$$\frac{dH}{dt} := \sum_i \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i = \sum_i \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) = 0.$$

By defining $y := (q_1, \ldots, q_k, p_1, \ldots, p_k)^T \in \mathbb{R}^{2k}$ ($2k = d$) we can rewrite the system in the compact form

$$\frac{dy}{dt} = J\nabla H(y), \qquad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Hamiltonian Flows and Invariant Measures

Let's call $\Phi_t : \mathbb{R}^{2k} \to \mathbb{R}^{2k}$ the flow generated by the Hamilton's equations

$$\dot{q}_i = \frac{\partial H(p,q)}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H(p,q)}{\partial q_i}$$

The Lebesgue measure ν on $\mathbb{R}^{2k}$ is invariant under the Hamiltonian flow Φ_t, that is for all measurable sets A

$$\nu(\Phi_t(A)) = \nu(A).$$

This is a consequence of

Theorem (Liouville's Theorem)

Let a vector field $f : \mathbb{R}^d \to \mathbb{R}^d$ be divergence free. If the flow map Φ_t satisfies

$$\frac{d}{dt}\Phi_t(y) = f(\Phi_t(y)),$$

then Φ_t is a volume preserving map for all t.

Finite Dimensional Gibbs Measure

A more interesting measure than the volume is the Gibbs measure. We have in fact:

Theorem (Invariance of Gibbs measures)

Assume that Φ_t is the flow generated by the Hamiltonian system above. Then the Gibbs measures defined as

$$d\mu := e^{-\beta H(p,q)} \prod_{i=1}^{d} dp_i \, dq_i$$

with $\beta > 0$, are invariant under the flow Φ_t.

The proof is trivial since from conservation of the Hamiltonian H the functions $e^{-\beta H(p,q)}$ remain constant, while, thanks to Liouville's Theorem the volume $\prod_{i=1}^{d} dp_i \, dq_i$ remains invariant as well.

Infinite Dimensional Hamiltonian systems

Consider the Cauchy problem

$$\begin{cases} (i\partial_t + \Delta)u = |u|^4 u \\ u(0,x) = u_0(x), \ \text{where } x \in \mathbb{T}. \end{cases}$$

with Hamiltonian

$$H(u(t)) = \frac{1}{2}\int |\nabla u|^2(x,t)\,dx + \frac{1}{3}\int |u(t,x)|^6\,dx.$$

One can rewrite the Cauchy problem as

$$\dot{u} = i\frac{\partial H(u,\bar{u})}{\partial \bar{u}}$$

and if we think of u as the infinite dimensional vector given by its Fourier coefficients $(\hat{u}(k))_{k\in\mathbb{Z}^n} = (a_k, b_k)_{k\in\mathbb{Z}^n}$, then this becomes an infinite dimensional Hamiltonian system.

Lebowitz, Rose and **Speer** considered the Gibbs measure *formally* given by

$$\text{“}d\mu = \exp\left(-\beta H(u)\right)\prod_{x\in\mathbb{T}} du(x)\text{”}$$

for $\beta > 0$ and showed that μ is a well-defined probability measure on $H^s(\mathbb{T})$ for any $s < \frac{1}{2}$.

The Gaussian Measure

How do we make sense of the Gibbs measure introduced above? We need to go through the Gaussian measure. Note that the quantity

$$H(u) + \frac{1}{2}\int |u|^2(x)\,dx$$

is also conserved. Then the best way to make sense of the Gibbs measure μ is by writing it as

$$\text{“}d\mu = \exp\left(-\frac{1}{6}\int |u|^6\,dx\right)\exp\left(-\frac{1}{2}\int (|u_x|^2 + |u|^2)\,dx\right)\prod_{x\in\mathbb{T}} du(x)\text{”}.$$

In this expression

$$\text{“}d\rho = \exp\left(-\frac{1}{2}\int (|u_x|^2 + |u|^2)\,dx\right)\prod_{x\in\mathbb{T}} du(x)\text{”}$$

is the Gaussian measure and

$$\frac{d\mu}{d\rho} = \exp\left(-\frac{1}{6}\int |u|^6\,dx\right),$$

corresponding to the nonlinear term of the Hamiltonian, is understood as the Radon-Nikodym derivative of μ with respect to ρ.

More on the Gaussian Measure

Our Gaussian measure ρ is defined as weak limit of the finite dimensional Gaussian measures

$$d\rho_N = Z_{0,N}^{-1} \exp\Big(-\frac{1}{2} \sum_{|k|\leq N} (1+|k|^2)|\widehat{v}_k|^2 \Big) \prod_{|k|\leq N} da_k db_k \, .$$

The measure ρ_N above can be regarded as the induced probability measure on $\mathbb{R}^{4N+2}$ under the map

$$\omega \longmapsto \Big\{ \frac{g_k(\omega)}{\sqrt{1+|k|^2}} \Big\}_{|k|\leq N} \quad \text{and} \quad \widehat{v}_k(\omega) = \frac{g_k(\omega)}{\sqrt{1+|k|^2}},$$

where $\{g_k(\omega)\}_{|k|\leq N}$ are independent identically distributed standard complex Gaussian random variables on a probability space $(\Omega, \mathcal{F}, P)$.

Invariance of the Gibbs Measure and Almost Surely Global Well-posedness

Theorem (Bourgain)

Consider the Cauchy problem

$$\begin{cases} (i\partial_t + \Delta)u = |u|^4 u \\ u(0,x) = \phi^\omega(x), \ \ \textit{where } x \in \mathbb{T}. \end{cases} \tag{5.1}$$

where

$$\phi^\omega(x) = \sum_{k\in\mathbb{Z}} \frac{g_k(\omega)}{\sqrt{1+|k|^2}} e^{k\cdot x}.$$

The data ϕ^ω are in the support of the Gibbs measure μ and there exists $\Omega \subset H^s$, $s = 1/2-$, such that $\mu(\Omega) = 1$ and for any $\phi^\omega \in \Omega$ the IVP (5.1) *is globally well-posed. Moreover μ is* **invariant**.

Remark

If one considers (5.1) *in the focusing case, then the theorem above holds if one imposes the restriction that the mass is small.*

Evolution Equations and Randomization

What is important about Bourgain's result?

- In the deterministic case one can only prove l.w.p in $H^s, s > 0$. So for the defocusing case g.w.p. in H^1 will follow from conservation of Hamiltonian. Bourgain uses the invariance of the measure to extend *almost surely* local solutions to global when $s = 1/2-$ and one has no conservation laws[3].
- Why does randomization help? Consider the randomized initial data

$$\phi^\omega(x) = \sum_{k\in\mathbb{Z}} \frac{g_k(\omega)}{\sqrt{1+|k|^2}} e^{k\cdot x}.$$

Although this initial data is in a rough space its linear flow $S(t)\phi^\omega(x)$ enjoys *almost surely* improved L^p bounds. These bounds yield improved nonlinear estimates *almost surely* arising in the analysis of

$$w(t,x) = u(t,x) - S(t)\phi^\omega(x),$$

where u is the solution of the equation at hand and as a consequence w solves a certain difference equation.

[3] Still small L^2 norm is needed in the focusing case

Randomization = Better estimates

The *almost sure* improved regularity mentioned above -akin to the role of Kintchine inequalities in Littlewood-Paley theory- stems from classical results of **Rademacher, Kolmogorov**, **Paley** and **Zygmund** proving that random series on the torus enjoy better L^p bounds than deterministic ones. For example, consider *Rademacher Series*

$$f(\tau) := \sum_{n=0}^{\infty} a_n\, r_n(\tau) \qquad \tau \in [0,1), \quad a_n \in \mathbb{C}$$

$$r_n(\tau) := \operatorname{sign} \sin(2^{n+1}\pi\,\tau)$$

Classical Theorem (Zygmund's book)

If $a_n \in \ell^2$ then the sum $f(\tau)$ belongs to $L^p([0,1))$ for all $p \geq 2$. More precisely,

$$\left(\int_0^1 |f|^p\, d\tau\right)^{1/p} \approx_p \|\, a_n\, \|_{\ell^2}$$

More on Evolution Equations and Randomization

- There is an equivalent result for $\mathbb{T}^2$ **rational** tori also due to Bourgain where again *counting lemmas* are used.
- The use of the invariance of the measure has limitations since in higher dimensions the support is on extremely rough spaces where the multilinear analysis needed to control the nonlinear terms of the equation is so far not possible. In higher dimensions usually a radial assumption is put in place.
- Randomization techniques have now been used with or without the help of the invariant measure in several contexts:
  - Schrödinger Equations: **Bourgain**, **Tzvetkov**, **Thomann-Tzevtkov**, **Colliander-Oh**, **Nahmod-Oh-Rey-Bellet-S**, **Nahmod-Rey-Bellet-Sheffield-S**, **Bourgain-Bulut**, **Nahmod-S** and **Bényi, Oh, Pocovnicu**.
  - KdV Equations: **Bourgain**, **Oh** and **Richards**.
  - NLW Equations: **Burq-Tzvetkov**, **de Suzzoni**, **Bourgain-Bulut**, **Luehrmann-Mendelson**, **Bényi, Oh, Pocovnicu** and **Pocovnicu**.
  - Benjamin-Ono Equations: **Deng** and **Deng-Tzvetkov-Visciglia**.
  - Navier-Stokes Equations: **Deng-Cui**, **Zhang-Fang** and **Nahmod-Pavlovic-S**.

The Non-Squeezing Theorem

We recall a version of **Gromov's** famous theorem:

Theorem (Finite Dimensional Non-squeezing)

Assume that Φ_t is the flow generated by a finite dimensional Hamiltonian system as recalled above and that Φ_t is also a symplectic map. Fix $y_0 \in \mathbb{R}^{2k}$ and let $B_r(y_0)$ be the ball in $\mathbb{R}^{2k}$ centered at y_0 and radius r. If

$$C_R(z_0) := \{y = (q_1, \ldots, q_k, p_1, \ldots, p_k) \in \mathbb{R}^{2k} / |q_i - z_0| \leq R\},$$

is a cylinder of radius R, and

$$\Phi_t(B_r(y_0)) \subset C_R(z_0),$$

it must be that $r \leq R$.

Can one generalize this theorem to the infinite dimensional setting given by a periodic dispersive equation?

The infinite dimensional Non-squeezing Theorem

Generalizing this kind of result in infinite dimensions has been a long time project of **Kuksin** who proved, roughly speaking, that compact perturbations of certain linear dispersive equations do indeed satisfy the non-squeezing theorem.

We consider the Cauchy problem

$$\begin{cases} (i\partial_t + \Delta)u = |u|^2 u \\ u(0,x) = u_0(x), \ \text{where } x \in \mathbb{T} \end{cases}$$

with Hamiltonian

$$H(u(t)) = \frac{1}{2}\int |\nabla u|^2(x,t)\,dx + \frac{1}{4}\int |u(t,x)|^4\,dx.$$

As noted above, one can rewrite the Cauchy problem as

$$\dot{u} = i\frac{\partial H(u,\bar{u})}{\partial \bar{u}}$$

and if we think of u as the infinite dimensional vector given by its Fourier coefficients $(\hat{u}(k))_{k\in\mathbb{Z}^n} = (a_k, b_k)_{k\in\mathbb{Z}^n}$, then this becomes an infinite dimensional Hamiltonian system.

The infinite dimensional Non-squeezing Theorem

We consider again the flow

$$\begin{cases} \dfrac{d}{dt}u = i\dfrac{\partial H(u,\bar{u})}{\partial \bar{u}} \\ u(0,x) = u_0(x), \ \text{where } \ x \in \mathbb{T} \end{cases}$$

Also in this case, using Strichartz estimates and the conservation of mass one can prove global well-posedness for data in L^2, see **Bourgain**. Hence we can define a global flow map

$$\Phi(t)u_0 := u(x,t).$$

It is easy to show that the L^2 space equipped with the form

$$\omega(f,g) = \langle if, g\rangle_{L^2}$$

is a symplectic space for the initial data of the cubic, defocusing NLS equation on $\mathbb{T}$ and its global flow $\Phi(t)$ is a symplectomorphism.

The cubic, periodic, defocusing nonlinear Schrödingier flow $\Phi(t)$ introduced above is not a compact linear perturbation, hence it is not covered by Kuksin's work. Nevertheless **Bourgain** proved the following theorem:

Theorem (Infinite Dimension Non-squeezing)

Assume that Φ_t is the flow generated by the cubic, periodic, defocusing NLS equation in L^2. If we identify L^2 with l^2 via Fourier transform, we let $B_r(y_0)$ be the ball in l^2 centered at $y_0 \in l^2$ and radius r,

$$C_R(z_0) := \{(a_n) \in l^2 / |a_i - z_0| \leq R\}$$

a cylinder of radius R and

$$\Phi_t(B_r(y_0)) \subset C_R(z_0),$$

at some time t, then it must be that $r \leq R$.

Idea of the Proof

The proof of this theorem is based on the following steps

- Use the projection operator P_N to project the Cauchy problem onto a finite dimensional Hamiltonian system.
- Use Gromov's Theorem.
- Show that the flow $\Phi_N(t)$ of the projected problem approximates well the flow $\Phi(t)$ of the original problem.

The third item is the most difficult to prove. The tools used are strong multilinear estimates based on the Strichartz estimates.

Remark

- *Bourgain's argument may not work for other kinds of dispersive equations. For example for the KdV problem, the lemma in Bourgain's work that gives the good approximation of the flow $\Phi(t)$ by $\Phi_N(t)$ does not hold. See* **Colliander-Keel-S-Takaoka-Tao***.*
- *What about symplectic flows that as for now are known to be defined only almost surely? The only result on this is an weak non-squeezing theorem due to* **Dana Mendelson***.*

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Some Open Problems

- Understand Strichartz estimates for irrational tori in a more direct way, so that one can have more flexibility when using them for example in the almost sure well-posedness.
- Improve theorems on weak turbulence. See recent related work of **Faou-Hani-Germain**.
- Understand ergodic structures associated to infinite dimension Hamiltonian flows.
- Finding effective ways to extend a.s. local well-posedness to global when no Gibbs measure can be used, in particular in higher dimensions. See for example **Burq-Tzvetkov** and **Pocovnicu** for the NLW, **Colliander-Oh** for NLS and **Pavlovic-Nahmod-S** for the 2D Navier-Stokes equation.
- Use probabilistic approaches to study properties of discrete versions of dispersive equations.
  (See **Chatterjee-Kirkpatrick** and **Chatterjee**).
- Find more robust arguments to understand the symplectic structures associated to certain dispersive flows and their relationships to flows defined almost surely.