Global Aspects of Poisson Geometry

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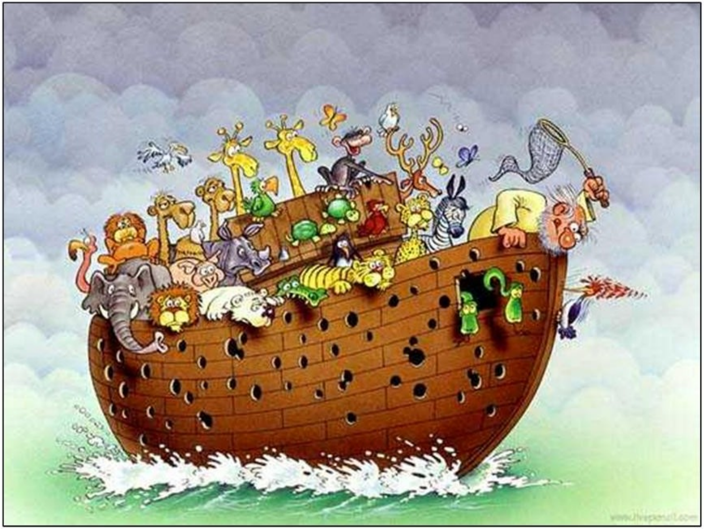

Aims of the talk:

- Tour of modern Poisson Geometry
- Some recent achievements
- Challenges/open problems

Classical Mechanics:

- Hamilton's formulation for the motion of a particle $q(t) \in \mathbb{R}^n$ in a potential $V : \mathbb{R}^n \to \mathbb{R}$:

$$m_i \ddot{q}_i(t) = -\frac{\partial V}{\partial q_i} \quad \Leftrightarrow \quad \begin{cases} \dot{q}_i = p_i \\ \dot{p}_i = -\frac{\partial V}{\partial q_i} \end{cases} \quad \Leftrightarrow \quad \dot{x}_a = \{h, x_a\}$$

w/ $(x_1, \dots, x_{2n}) = (q_1, \dots, q_n, p_1, \dots, p_n)$, $h = \sum_{i=1}^n \frac{p_i^2}{2m_i} + V$, and $\{\cdot, \cdot\}$ is the Poisson bracket:

$$\{f_1, f_2\} = \sum_{i=1}^n \left(\frac{\partial f_1}{\partial p_i} \frac{\partial f_2}{\partial q_i} - \frac{\partial f_1}{\partial q_i} \frac{\partial f_2}{\partial p_i} \right).$$

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Elasticity:

- Euler's Equation for the motion of a top in absence of gravity, moving around its center of mass, with moments of inertia I_1, I_2 and I_3:

$$\begin{cases} \dot{x}_1 = \frac{I_2 - I_3}{I_2 I_3} x_2 x_3 \\ \dot{x}_2 = \frac{I_3 - I_1}{I_3 I_1} x_3 x_1, \\ \dot{x}_3 = \frac{I_1 - I_2}{I_1 I_2} x_1 x_2. \end{cases} \qquad \Leftrightarrow \qquad \dot{x}_a = \{h, x_a\}$$

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Population Dynamics:

- The equations of Lotka-Volterra for the dynamics of the populations $(x_1, \ldots, x_n)$ of n biological species interacting in a closed ecosystem:

$$\dot{x}_i = \varepsilon_i x^i + \sum_{j=1}^{n} a_{ij} x_i x_j, \quad \Leftrightarrow \quad \dot{x}_i = \{h, x_i\}$$

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Definition

A Poisson bracket on an associative k-algebra $\mathcal{A}$ is a **Lie bracket** $\{\cdot,\cdot\}:\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ satisfying the **Leibniz identity**:

$$\{a_1, a_2a_3\} = \{a_1, a_2\}a_3 + a_2\{a_1, a_3\}.$$

Definition

A Poisson manifold is a manifold M together with a Poisson bracket on the algebra of smooth functions $\mathcal{A} = C^\infty(M)$.

In the previous examples: $M = \mathbb{R}^{2n}$, $\mathbb{R}^3$, $\mathbb{R}^n_+$.

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On a Poisson manifold $(M, \{\cdot,\cdot\})$ each $h \in C^\infty(M)$ determines a hamiltonian vector field X_h by:

$$X_h(f) := \{h, f\}, \quad \forall f \in C^\infty(M).$$

Basic Properties

- I is a first integral of X_h if and only if $\{h, I\} = 0$;
- h is always a first integral of X_h;
- If I_1 and I_2 are first integrals of X_h, then $\{I_1, I_2\}$ is also a first integral of X_h.

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(M, ω) a **symplectic manifold**: $\omega \in \Omega^2(M)$, with $\mathrm{d}\omega = 0$, inducing a non-degenerate pairing:

$$\omega \in TM \times TM \to \mathbb{R},$$

Then there is an induced Poisson bracket:

$$\{f_1, f_2\} := \pi(\mathrm{d}f_1, \mathrm{d}f_2),$$

where $\pi = \omega^{-1} : T^*M \times T^*M \to \mathbb{R}$.

Example

When $M = \mathbb{R}^{2n}$, $\omega = \sum_{i=1}^n \mathrm{d}p_i \wedge \mathrm{d}q_i$, then $\pi = \sum_{i=1}^n \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}$ and:

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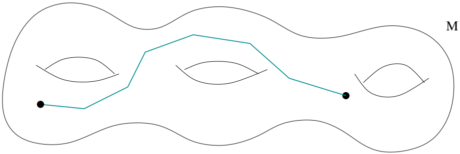
For any two hamiltonian functions h_1 and h_2:

$$[X_{h_1}, X_{h_2}] = X_{\{h_1, h_2\}}$$

Define an **equivalence relation** on M by declaring two points equivalent if they can be joined by trajectories of hamiltonian vector fields.



Theorem (Weinstein, 1983)

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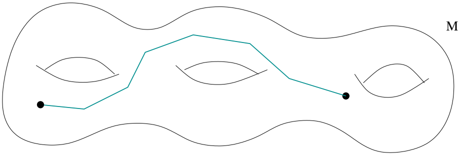

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Dimension 1: A curve M^1 only admits the zero Poisson bracket.

Dimension 2: On an (oriented) surface M^2 a Poisson bracket takes the form

$$\{f_1, f_2\} = g\,\mu^{-1}(\mathrm{d}f_1, \mathrm{d}f_2).$$

for some smooth function $g \in C^\infty(M^2)$ and non-vanishing 2-form $\mu \in \Omega^2(M)$. The symplectic leaves are:

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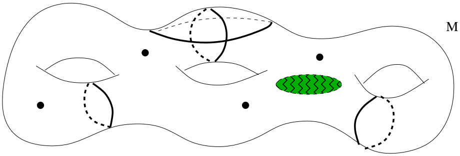
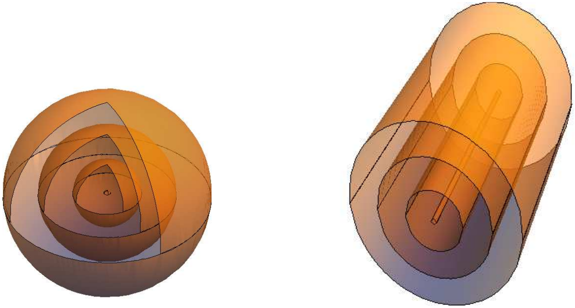


Dimension 3: On a 3-manifold a Poisson bracket partitions M^3 into symplectic leaves of dimension 2, and symplectic leaves of dimension 0 (can be isolated, or not!).



If M^3 is oriented, a non-vanishing 3-form $\mu \in \Omega^3(M)$ and a smooth function $g \in C^\infty(M^2)$ determine a Poisson bracket:

$$\{f_1, f_2\} = \mu^{-1}(\mathrm{d}g, \mathrm{d}f_1, \mathrm{d}f_2).$$

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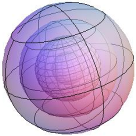
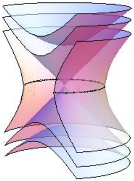
For example, if g is a **Morse function**, two types of singular level sets:

index 0 and 3  *index 1 and 2* 

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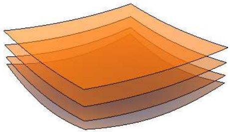

no local invariants

Poisson brackets are *too* general. Often one studies specific classes of Poisson structures.

- Linear Poisson structures: $M = V$ is a vector space and Poisson bracket of linear functions is a linear function:
  - $V = \mathfrak{g}^*$ for a Lie algebra $\mathfrak{g}$;
  - symplectic leaves=coadjoint orbits.

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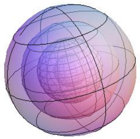

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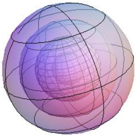

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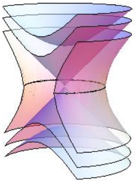

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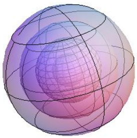

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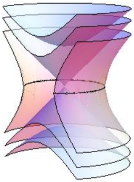

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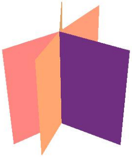
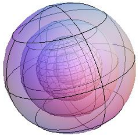
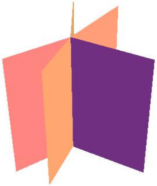
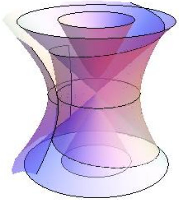


$\mathfrak{h}$

- Regular Poisson structures: Poisson structures whose symplectic leaves all have the same dimension, also known as **symplectic foliations**.
- Linear Poisson structures: Poisson structures on a vector space ($M = V$) for which the bracket of linear functions is a linear function.
- Multiplicative Poisson structures: Poisson structures on a Lie group ($M = G$) for which the multiplication preserves the Poisson bracket.
- b-symplectic structures: Poisson structures with non-degenerate (generic) singularities.

Points were the Poisson bracket vanishes are singularities of the Poisson structure:

At a singular point x_0 the tangent space carries a linear Poisson bracket, so $T_{x_0}M = \mathfrak{g}^*_{x_0}$. One calls $\mathfrak{g}_{x_0}$ the isotropy Lie algebra at x_0.

Theorem (Conn, 1985)

Let $(M, \{\ ,\ \})$ be a Poisson manifold and x_0 a zero of the Poisson bracket. If the isotropy Lie algebra $\mathfrak{g}_{x_0}$ is compact semisimple then the Poisson bracket can be linearized: there are local coordinates $(x^1, \ldots, x^m)$ centered at x_0 where the Poisson bracket is linear:

$$\{x^i, x^j\} = \sum_k c_k^{ij} x^k.$$

Remark: The original proof used a Nash-Moser fast convergence method, requiring some hard analysis. A (more soft) geometric proof was obtain in 2011 by M. Crainic & RLF.

Problem

Main issues in local Poisson geometry:

- *When can a Poisson bracket be linearized around a zero?*
- *Is there a local normal form for a given Poisson bracket?*
- *What are the local invariants of a Poisson bracket?*

Many people have worked on these questions since they were raised by A. Weinstein in a seminal paper in 1983:

- A. Alekseev, V.I. Arnold, G. Belitskii, K. Bhaskara, P. Dazord, J.P. Dufour, V. Ginzburg, J. Martinet, P. Monnier, N.-T. Zung, . . .

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Local Poisson Geometry

- Why is local Poisson geometry so difficult?

A Poisson bracket $(M, \{\cdot,\cdot\})$ in local coordinates $(U, x_1, \ldots, x_n)$ defines smooth functions $\pi_{ij} \in C^\infty(U)$:

$$\pi_{ij}(x) := \{x_i, x_j\}(x), \quad (i, j = 1, \ldots, n).$$

Skew-symmetry of the bracket gives $\pi_{ij} = -\pi_{ji}$, while the Jacobi identity amounts to:

$$\sum_{l=1}^{n} \left(\pi_{il} \frac{\partial \pi_{jk}}{\partial x_l} + \pi_{jl} \frac{\partial \pi_{ki}}{\partial x_l} + \pi_{kl} \frac{\partial \pi_{ij}}{\partial x_l} \right) = 0, \quad (1 \le i < j < k \le n).$$

For $n \geq 5$ this is an overdetermined system of non-linear PDEs.

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Global Poisson Geometry

Dealing with global issues in Poisson geometry (*Poisson topology?*) is an even more delicate matter.

- Examples of good questions in global Poisson geometry:
  - **Normal forms/rigidity:** Is there a normal form for a Poisson bracket around a symplectic leaf?
  - **Deformation:** Can two nearby Poisson brackets be deformed, one to the other, through a small deformation? If not, can one describe all nearby Poisson structures?
  - **Stability of leafs:** Is a leaf of a Poisson bracket structurally stable? Does any nearby Poisson structure have a nearby diffeomorphic/symplectomorphic leaf?

Global Poisson Geometry

- Examples of bad questions in global Poisson geometry:
  - **Existence:** Does a manifold admit a Poisson structure?
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Remark: *Bad* questions can be turned to *good* questions by restricting to appropriate classes of Poisson manifolds.

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Remark: *Bad* questions can be turned to *good* questions by restricting to appropriate classes of Poisson manifolds.

A Poisson bracket makes $(C^\infty(M), \{\cdot,\cdot\})$ into a Lie algebra.

- **Question:** Is there a Lie group "integrating" $(M, \{\cdot,\cdot\})$?

Such a Lie group, if it exists, should play a fundamental role in global Poisson geometry. Amazingly, the answer is even better:

- **Answer:** (M. Karasev; A. Weinstein) There is a group-like object, a symplectic groupoid, associated with every Poisson manifold $(M, \{\cdot,\cdot\})$.

But there are no free meals...

- **Addenda:** (M.Crainic & RLF) This object always exists as a topological groupoid, is finite dimensional, but may fail to be smooth. The precise obstructions to smoothness are known.

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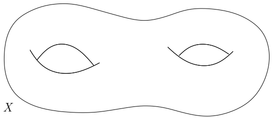
- **Answer:** (M. Karasev; A. Weinstein) There is a group-like object, a symplectic groupoid, associated with every Poisson manifold $(M, \{\cdot,\cdot\})$.

But there are no free meals...

- **Addenda:** (M.Crainic & RLF) This object always exists as a topological groupoid, is finite dimensional, but may fail to be smooth. The precise obstructions to smoothness are known.

Digression into basic topology

X – topological space; look at paths $\gamma : [0, 1] \to X$



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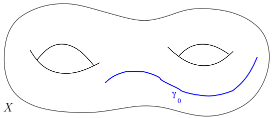

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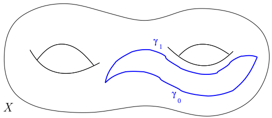

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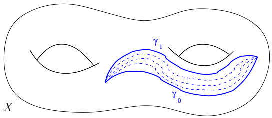

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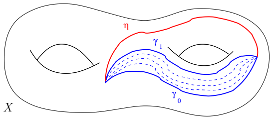

$$\Pi_1(X) = \{[\gamma] \mid \gamma : [0,1] \to X\}$$

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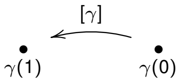

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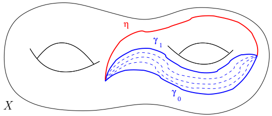

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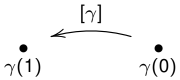
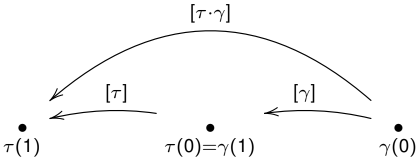


- **product**:



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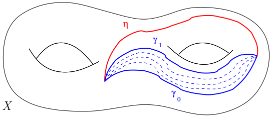

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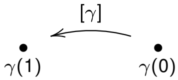

- **identity**:

$$u : X \hookrightarrow \Pi_1(X)$$

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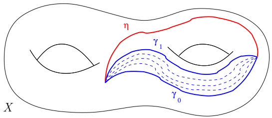

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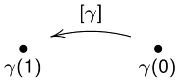
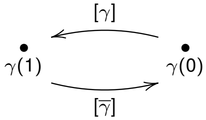


- **inverse**:

$$\iota : G \longrightarrow G$$



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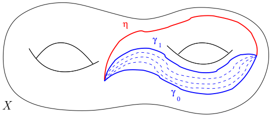

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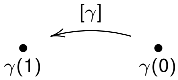

- The space $\Pi_1(X)$ has a natural topology and the source, target, multiplication and inverse are all continous maps: $\Pi_1(X) \rightrightarrows X$ is an example of a topological groupoid.

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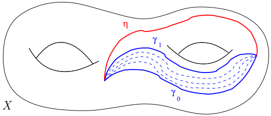

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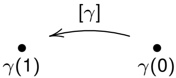

- The space $\Pi_1(X)$ has a natural topology and the source, target, multiplication and inverse are all continous maps: $\Pi_1(X) \rightrightarrows X$ is an example of a topological groupoid.
- If $X = M$ is a manifold, the space $\Pi_1(M)$ is a manifold and the source, target, multiplication and inverse are all smooth maps: then $\Pi_1(M) \rightrightarrows M$ is an example of a Lie groupoid.

$(M, \{\cdot,\cdot\})$ – Poisson manifold; look at cotangent paths:

$$[0,1] \xrightarrow{a} T^*M$$

- For any Poisson manifold $(M, \{\cdot,\cdot\})$, there is a topological groupoid $\Sigma(M) \rightrightarrows M$ "integrating" it.
- $\Sigma(M) = P(T^*M)//G$ is a symplectic quotient (A. Cattaneo & G. Felder).

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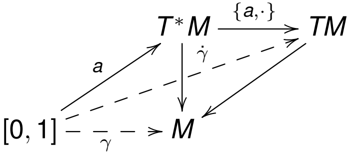

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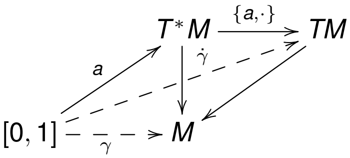

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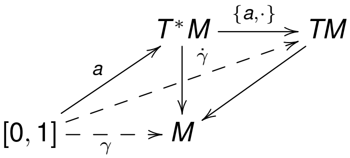

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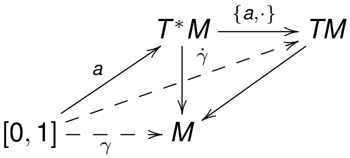

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$\Sigma(M) \rightrightarrows M$ and $\Pi_1(M) \rightrightarrows M$ differ substantially:

- $\Pi_1(M) \rightrightarrows M$ is always smooth while $\Sigma(M) \rightrightarrows M$ may fail to be smooth;
- $\Pi_1(M)$ has one orbit (if M connected) while orbits of $\Sigma(M)$ are the symplectic leaves of $(M, \{\cdot,\cdot\})$;
- The homotopy groups

$$\pi_1(M,x) = \frac{\{\text{loops in } M \text{ based at } x\}}{\text{homotopy}}$$

are *discrete* while the Poisson homotopy groups

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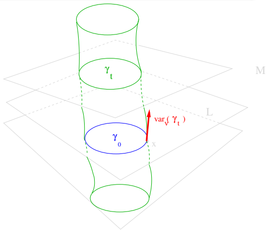
Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold and fix a symplectic leaf L and $x \in L$. There is a group morphism

$$\partial_x : \pi_2(L, x) \to \nu_x^*(L)$$

controlling integrability: $\Sigma(M)$ is smooth if and only if the groups $Im(\partial_x)$ are uniformly discrete.

$$\langle \partial_x(\gamma_0), \mathrm{var}_\nu(\gamma_t)\rangle = \frac{d}{dt}\int_{\mathbb{S}^2} \gamma_t^* \omega_L \Big|_{x=0}$$



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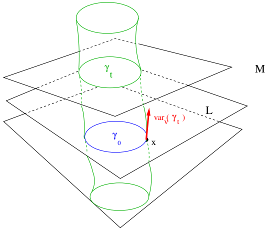

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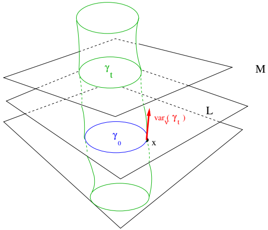

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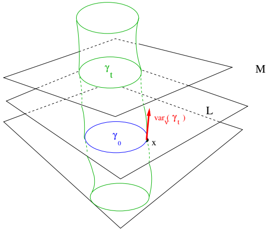

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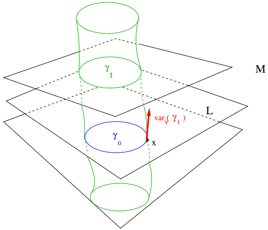

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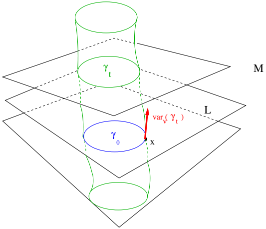

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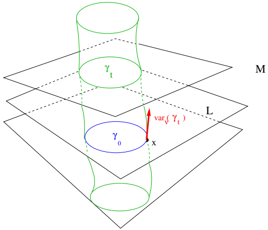

Theorem (Crainic & Marcut (2012), del Hoyo & RLF (2015))

Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold. If $\Sigma(M, x)$ is smooth and the source map is proper, then a neighborhood of any symplectic leaf L is Poisson diffeomorphic to the first order model of $\{\cdot,\cdot\}$ around L.

- There is an explicit local model, which depends on some choices.
- This result can be strengthen by replacing $\Sigma(M)$ by other symplectic groupoids integrating $(M, \{\cdot,\cdot\})$
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A consequence of these type of results is various forms of "rigidity" for Poisson brackets. For example:

Corollary (Marcut, 2013)

Let $(\mathbb{S}(\mathfrak{g}^*), \{\cdot,\cdot\}_0)$ *be the unit sphere in the dual of a compact semisimple Lie algebra* $\mathfrak{g}$. *In the space of Poisson structures there is an open neighborhood* $\mathcal{V}$ *of* $\{\cdot,\cdot\}_0$ *such that every Poisson structure in* $\mathcal{V}$ *is isomorphic to* $f\{\cdot,\cdot\}_0$, *where* f *is a smooth function constant on the symplectic leaves.*

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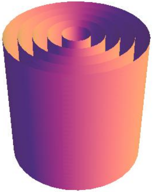

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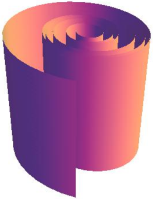

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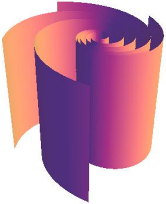

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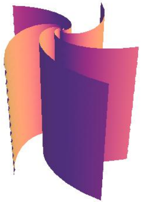

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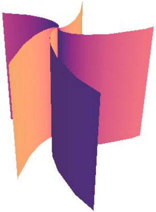

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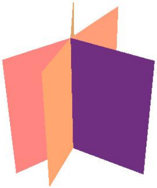

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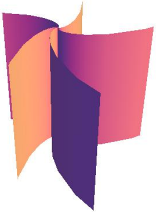

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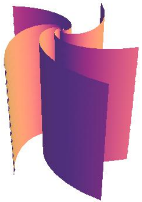

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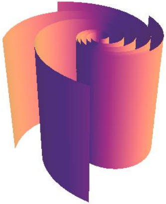

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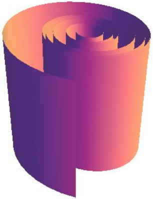

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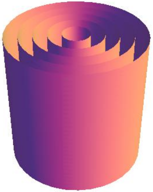

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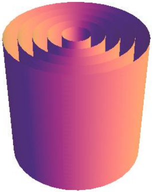

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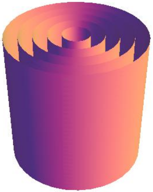

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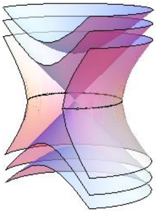

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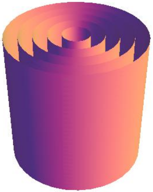

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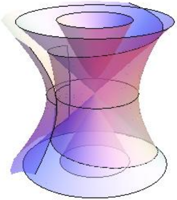

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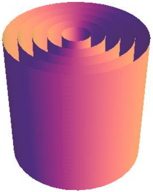

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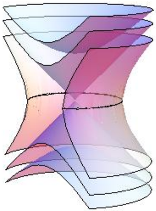

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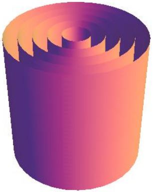

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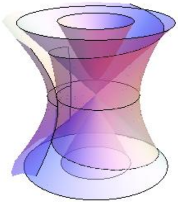

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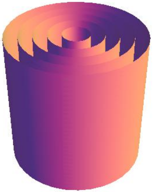

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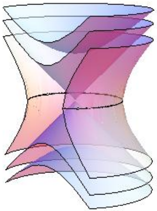

Theorem (M. Crainic & RLF (2010))

Let L be a compact symplectic leaf of a Poisson manifold $(M, \{\cdot,\cdot\})$ *and assume that*

$$H^2_\pi(M, L) = 0.$$

Then L is stable: every nearby Poisson structure has a family of nearby diffeomorphic leaves smoothly parametrized by $H^1_\pi(M, L)$*.*

- The relative Poisson cohomology $H^\bullet_{\pi,L}(M)$ is the cohomology of the complex of multivector fields along L with a differential induced from $\{\cdot,\cdot\}$ (an *elliptic complex*).
- There is also a version for *strong stability* where "diffeomorphic" is replaced by "symplectomorphic".
- The proofs involve some ideas on deforming linear complexes to *non-linear* complexes, that can be traced back to unpublished work of Hamilton (deformations of foliations).

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Poisson manifolds M which integrate to a proper symplectic groupoid play a role in Poisson geometry analogous to the role of compact Lie groups in Lie theory.

Some amazing properties (N.T. Zung, Crainic-RLF-Torres):

- The leaf space is an integral affine orbifold;
- The symplectic form varies linearly;
- They possess canonical Duistermaat-Heckman type measures;
- They have an associated *symplectic gerbe* with a "lagrangian" Dixmier-Douady class in $\check{H}^2(B, \mathcal{T}_{lag})$;
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Existence Problem

Notice that:

1. any manifold M admits the zero Poisson structure, and
2. if $\dim M \geq 2$, given sets $U \subset K \subset M$, with U open and K compact, there are Poisson brackets on M which do not vanish on U and have support contained in K.

The problem of existence of Poisson structures on a given manifold only makes sense if one puts additional restrictions.

- Given a manifold M does it admit a *regular* Poisson structure (=a symplectic foliation) of a fixed codimension q?

1. If M is open, a version of Gromov's h-principle holds: existence holds iff there exists a foliation of codimension q admitting a leafwise non-degenerate 2-form (P. Frejlich & RLF, 2012);
2. if M is closed, the problem is wide open and only very partial results are known:
   - It is not known if an odd sphere $\mathbb{S}^{2d+1}$ admits a codimension 1 symplectic foliation, except for the 3-sphere (the Reeb foliation) and the 5-sphere (Y. Mitsumatsu, unpublished)
   - There is a recent PhD Thesis (unpublished) about existence of codimension 1 symplectic foliations on manifolds $M = N \times \mathbb{S}^1$ (B. Osorno-Torres).

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- Poisson structures with generic singularities (= *b*-symplectic manifolds): Guillemin, Miranda & Pires; Gualtieri, Pelayo & Ratiu; Marcut & Osorno-Torres ...
- Generalized Geometry: Hitchin; Bursztyn, Calvacanti & Gualtieri, Baley; ...
- Poisson-Lie groups and Poisson homogeneous spaces: Drinfeld; Semenov-Tian-Shansky; Lu & Evens; Yakimov; Kosmann-Schwarzbach; Reshetikhin ...
- Moduli spaces and twisted-Poisson structures: Alekseev & Meinrenken; Boalch; Li-Bland & Severa; ...
- Cluster algebras: Fomin & Zelevinsky; Gekhtman, Shapiro & Vainshtein; ...

and zillions of applications ...

. . . and there is still a lot of very tasty *Poisson* to be fished!!!

http://poissongeometry.org