Colored partitions of a convex polygon by non-intersecting diagonals[1]

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(joint work with D. Birmajer and J. Gil)

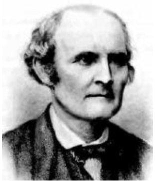
[1] Preprint arXiv:1503.05242, March 2015

Over one hundred years ago Arthur Cayley considered the following problem:

"The partitions are made by non-intersecting diagonals; the problem which have been successively considered are

1. to find the number of partitions of an r-gon into triangles,
2. to find the number of partitions of an r-gon into k parts, and
3. to find the number of partitions of an r-gon into p-gons, r of the form $n(p-2)+2$."

[Cayley, On the partition of a polygon, March 12, 1891]



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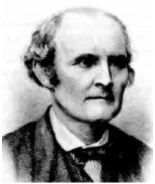
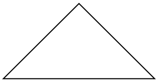
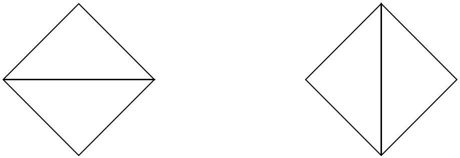

Arthur Cayley
(1821-1895)

Triangulations of an $(r+2)-gon$

$r = 1$: a triangle can be partitioned into exactly one triangle.

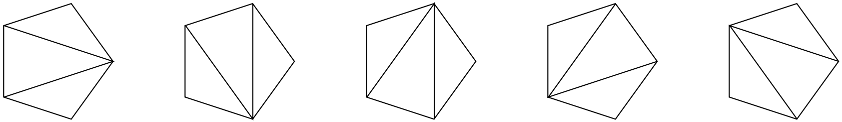


$r = 2$: a quadrilateral can be partitioned into two triangles in two different ways.



Triangulations of an $(r+2)-gon$

$r = 3$: a pentagon can be partitioned into three triangles in five different ways.



Triangulations of an $(r+2)-gon$

And so on...

Table : Number of Dissections

$r =$	1	2	3	4	5	6	7	8	9
$\hat{p}(r,r) =$	1	2	5	14	42	132	429	1430	4862

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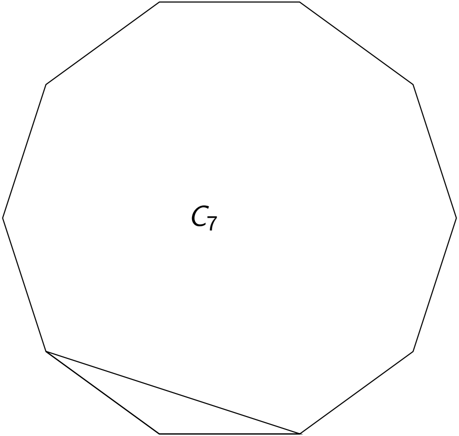
These are the Catalan numbers (A000108):

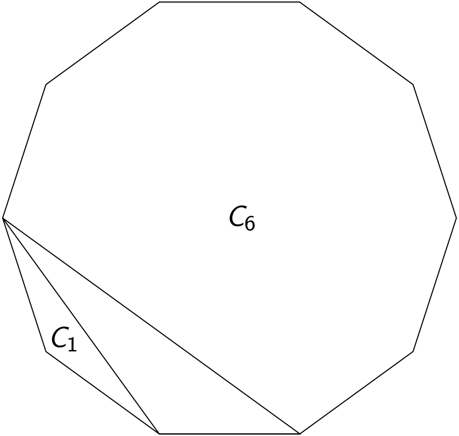
$$C_n = \frac{1}{n+1}\binom{2n}{n}, \qquad C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$

Eugene Charles Catalan (1814-1894)

$$C_8 = C_0 C_7 + \dots$$



$$C_8 = C_0 C_7 + C_1 C_6 + \dots$$



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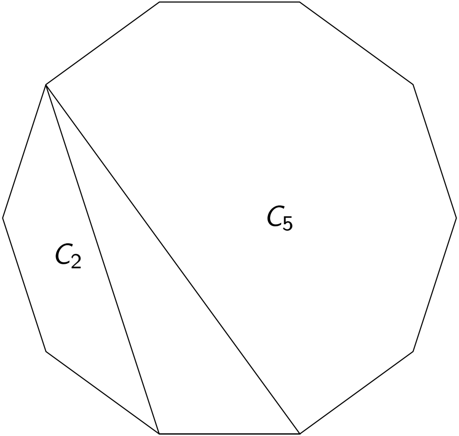
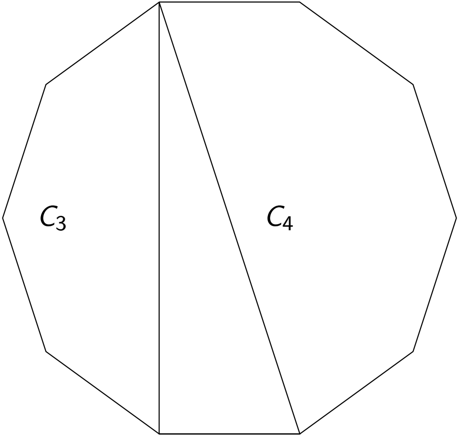


$$C_8 = C_0 C_7 + C_1 C_6 + C_2 C_5 + C_3 C_4 \dots$$



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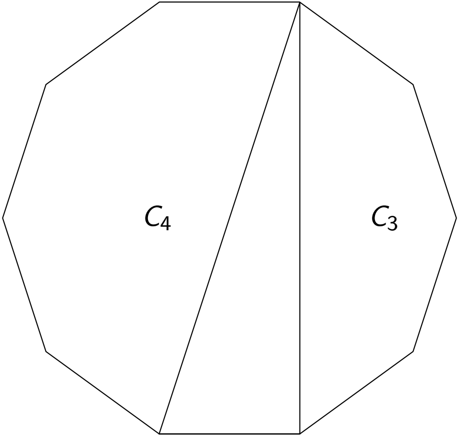

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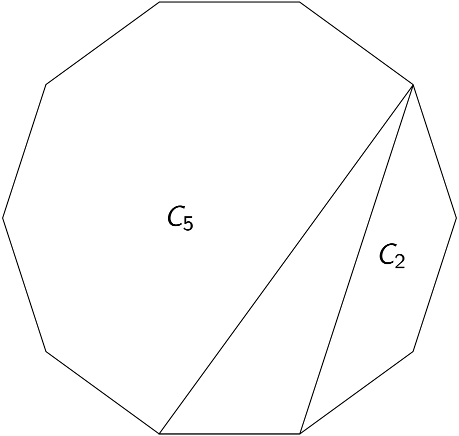

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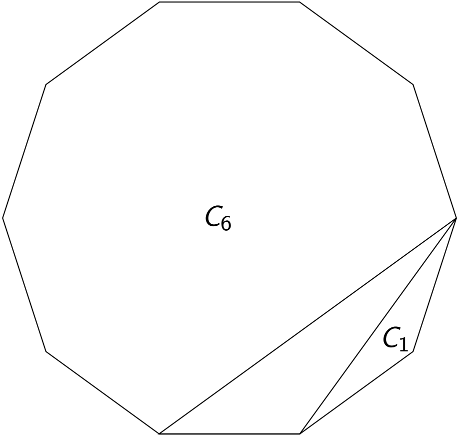

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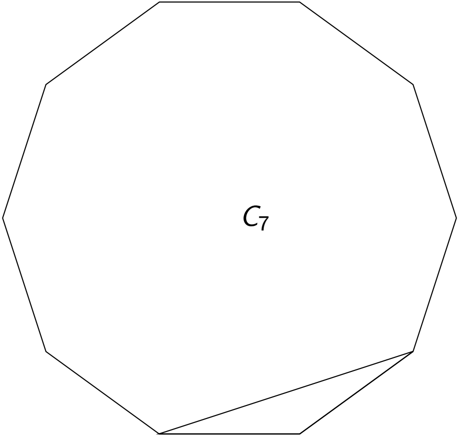

A binary tree is a tree data structure in which each node has at most two children. Let's try to count the number of rooted binary trees with $n+1$ total nodes.

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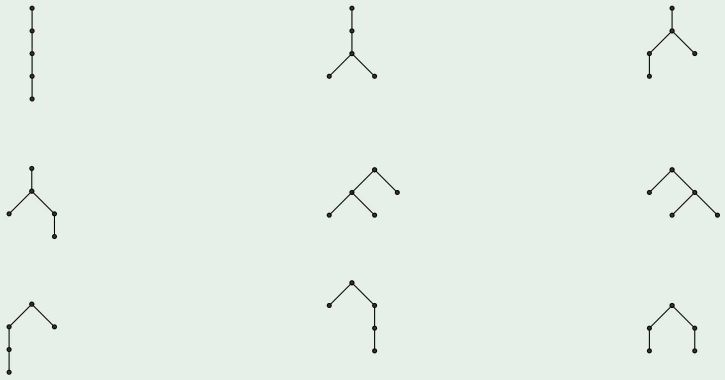
Example ($n = 4$, $t(4) = 9$)



Rooted binary trees with $(n+1)$ nodes

Table : Number of Trees

$n =$	0	1	2	3	4	5	6	7	8
$t(n) =$	1	1	2	4	9	21	51	127	323

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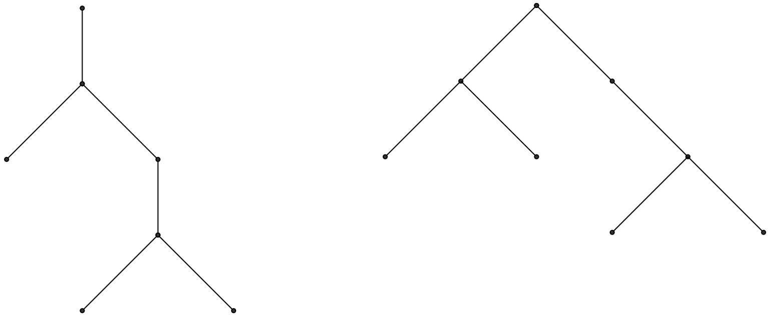
These are the Motzkin numbers (A001006):

$$M_{n+2} = M_{n+1} + \sum_{i=0}^{n} M_i M_{n-i}$$

Theodore Motzkin (1908-1970)



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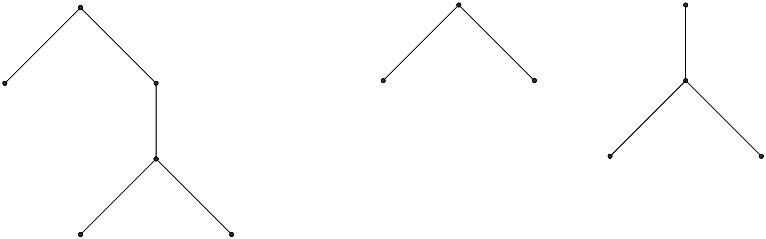

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Faà di Bruno

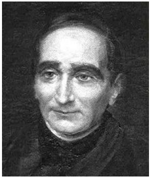
Faà di Bruno's formula stated in terms of partial Bell polynomials[2]:

Suppose

$$f(t) = \sum_{n=1}^{\infty} a_n t^n \qquad \text{and} \qquad g(t) = \sum_{n=0}^{\infty} b_n t^n.$$

Then

$$g(f(t)) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} (k! b_k) B_{n,k}(1! a_1, 2! a_2, \dots) \right] \frac{t^n}{n!}.$$



Francesco Faà di Bruno (1825-1888)

[2] The name "Bell polynomials" was introduced by Riordan in *An Introduction to Combinatorial Analysis*, 1958

Consequently, Bell polynomials are useful to work with Lagrange Inversion.

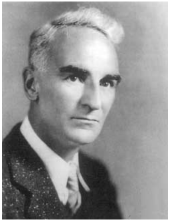
Partial Bell Polynomials

$$B_{n,k}(x_1,\dots,x_{n-k+1}) = \sum_{\alpha} \frac{n!}{\alpha_1!\alpha_2!\cdots\alpha_{n-k+1}!} \prod_{\ell=1}^{n-k+1} \left(\frac{x_\ell}{\ell!}\right)^{\alpha_\ell},$$

where the sum runs over all multi-indices

$\alpha \in \mathbb{N}_0^{n-k+1}$ such that

$$\alpha_1 + \alpha_2 + \cdots = k \quad \text{and} \quad \alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots = n.$$



Eric Temple Bell
(1883-1960)

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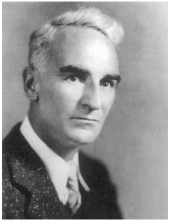
Complete Bell Polynomials (Exponential Polynomials):

$$B_n(x_1,\dots,x_n) = \sum_{k=1}^{n} B_{n,k}(x_1,\dots,x_{n-k+1}).$$



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Combinatorial Meaning

Example

$$B_{4,2}(x_1, x_2, x_3) = 4x_1x_3 + 3x_2^2$$

looks at partitions of a four element set $\{a, b, c, d\}$ into two parts:

$\{a, b, c\}, \{d\}$ $\{a, b, d\}, \{c\}$ $\{a, c, d\}, \{b\}$ $\{b, c, d\}, \{a\}$

$\{a, b\}, \{c, d\}$ $\{a, c\}, \{b, d\}$ $\{a, d\}, \{b, c\}$

Bell Identities

$$B_{n,0}(x_1, x_2, \dots) = 0 \quad \text{except} \quad B_{0,0}(x_1, x_2, \dots) = 1,$$

$$B_{n,k}(x_1, 0, 0, \dots) = 0 \quad \text{except} \quad B_{n,n}(x_1, 0, 0, \dots) = x_1^n,$$

$$B_{n,k}(\tfrac{x_2}{2}, \tfrac{x_3}{3}, \dots) = \frac{n!}{(n+k)!} B_{n+k,k}(0, x_2, x_3, \dots),$$

$$B_{n,k}(0, \dots, 0, x_j, 0, \dots) = 0, \quad \text{except} \quad B_{jk,k} = \frac{(jk)!}{k!(j!)^k}\, x_j^k,$$

$$B_{n,k}(x_1 + x_1', x_2 + x_2', \dots) = \sum_{\substack{\kappa \le k \\ \nu \le n}} \binom{n}{\nu} B_{\nu,\kappa}(x_1, x_2, \dots) B_{n-\nu,k-\kappa}(x_1', x_2', \dots).$$

Specific Partial Bell Values

$$B_{n,k}(1,1,1,\dots) = S(n,k), \quad \text{(Stirling number of the 2nd kind)}$$

$$B_{n,k}(1!,2!,3!,\dots) = \binom{n-1}{k-1}\frac{n!}{k!}, \quad \text{(Lah number)}$$

$$B_{n,k}(0!,1!,2!,\dots) = |s(n,k)|, \quad \text{(signless Stirling number of the 1st kind)}$$

$$B_{n,k}(1,2,3,\dots) = \binom{n}{k}k^{n-k}, \quad \text{(idempotent number)}$$

$$B_{n,k}(c_1,2c_2,0,0,\dots) = \frac{n!}{k!}\binom{k}{n-k}c_1^{2k-n}c_2^{n-k}.$$

Convolution Formulas

For any sequence (c_j), and $r, k \in \mathbb{N}$, define

$$q(r,k) = \frac{1}{(r-k+1)!} B_{r,k}(1!c_1, 2!c_2, \dots),$$

and let

$$y_r = \sum_{k=0}^{r} q(r,k).$$

Theorem

For $r, k \in \mathbb{N}$,

$$\sum_{\ell=1}^{k-1} \sum_{m=\ell}^{r-1} q(r-m, k-\ell) q(m,\ell) = \frac{2(k-1)}{r-k+2} q(r,k).$$

Convolution Formulas

Corollary

Similarly, for any $d, r, k \in \mathbb{N}$,

$$\sum_{\substack{\ell_1+\cdots+\ell_d=k \\ m_1+\cdots+m_d=r}} q(m_1,\ell_1)\ldots q(m_d,\ell_d) = d\,\frac{\binom{r+d-1}{d-1}}{\binom{r-k+d}{d-1}}\,q(r,k)$$

and

$$\sum_{m_1+\cdots+m_d=r} y_{m_1}\cdots y_{m_d} = d\sum_{i=1}^{r}\binom{r+d-1}{i-1}\frac{(i-1)!}{r!}B_{r,i}(\mathbf{c})$$

Convolution Formulas

Example

Choosing $c_j = 1$ for all j, gives $y_r = C_r$ which, with $d = 2$, gives the convolution formula

$$\sum_{i=0}^{r} y_i y_{r-i} = y_{r+1}.$$

Similarly, choosing $c_1 = 1$, $c_2 = 1$ and $c_j = 0$ for all $j > 2$, gives $y_r = M_r$ which, with $d = 2$, gives the convolution formula

$$\sum_{i=0}^{r} y_i y_{r-i} = y_{r+2} - y_{r+1}.$$

Convolution Formulas

For any a and b, and any sequence (c_j), define

$$q(r,k) = \frac{\binom{ar+(b-1)k+1}{k}}{ar+(b-1)k+1} \cdot \frac{k!}{r!} B_{r,k}(1!c_1, 2!c_2, \dots),$$

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For $r, k \in \mathbb{N}$,

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Convolution Formulas

Corollary (BGW[3])

Similarly, for any $d, r, k \in \mathbb{N}$,

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and

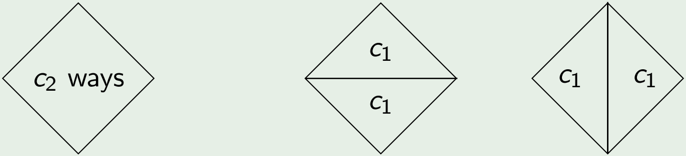
$$\sum_{m_1+\cdots+m_d=r} y_{m_1}\cdots y_{m_d} = d\sum_{i=0}^{r}\frac{\binom{ar+(b-1)i+d}{i}}{ar+(b-1)i+d}\cdot\frac{i!}{r!}B_{r,i}(\mathbf{c})$$

[3] Birmajer, Gil and Weiner, *Some convolution identities and an inverse relation involving partial Bell polynomials*, Electron. J. Combin. **19** (2012), no. 4, Paper 34, 14 pp.

Colored partitions of $(r+2)$-gon into k parts

For $\mathbf{c} = (c_1, c_2, \dots)$ with $c_j \in \mathbb{N}_0$, we are interested in the set $\mathscr{P}^{\mathbf{c}}(r,k)$ of colored partitions of a convex $(r+2)$-gon made by $k-1$ non-intersecting diagonals into $(j+2)$-gons colored in c_j possible ways.

Example ($r = 2$, quadrilaterals)



$|\mathscr{P}^{\mathbf{c}}(2,1)| = c_2$ $|\mathscr{P}^{\mathbf{c}}(2,2)| = 2c_1^2.$

Remember $\sum_{\ell=1}^{k-1}\sum_{m=\ell}^{r-1} q(r-m,k-\ell)q(m,\ell) = \frac{2(k-1)}{ar+(b-2)k+2}\, q(r,k)$?

Theorem

Given $\mathbf{c}$, *and* $k>0$, *let* $\hat{p}(r,k) = \left|\mathscr{P}^{\mathbf{c}}(r,k)\right|$. *Then*

$$\sum_{\ell=1}^{k-1}\sum_{m=\ell}^{r-1} \hat{p}(r-m,k-\ell)\hat{p}(m,\ell) = \frac{2(k-1)}{r+2}\hat{p}(r,k),$$

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or equivalently,

$$(r+2)\sum_{\ell=1}^{k-1}\sum_{m=\ell}^{r-1} \hat{p}(r-m,k-\ell)\hat{p}(m,\ell) = 2(k-1)\hat{p}(r,k).$$

Colored partitions of $(r+2)$-gon into k parts

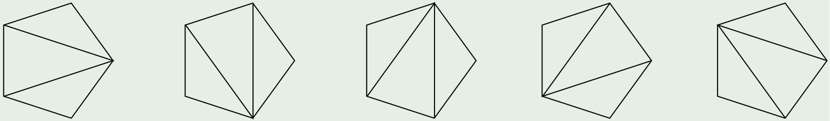
Proof: Let $k > 1$ and consider the set $\mathscr{S}^{\mathbf{c}}(r,k)$ of colored partitions in $\mathscr{P}^{\mathbf{c}}(r,k)$ with the additional structure of a specified diagonal with one of its vertices labeled.

Colored partitions of $(r+2)$-gon into k parts

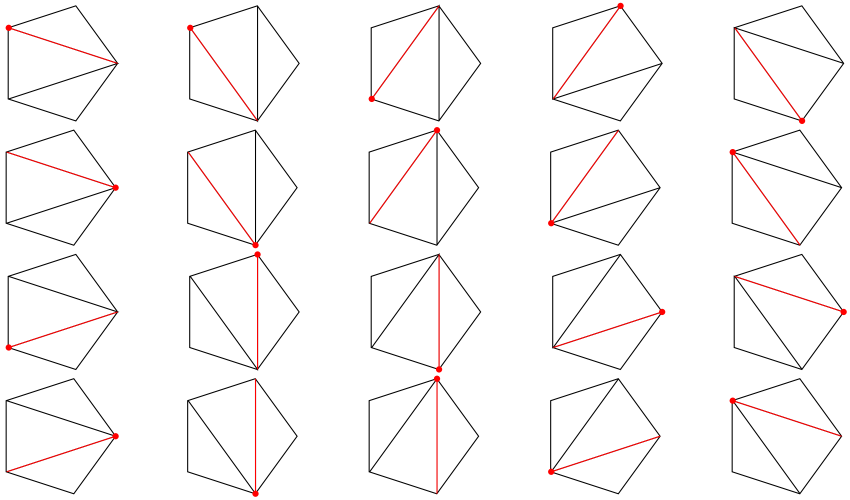
Proof: Let $k > 1$ and consider the set $\mathscr{S}^{\mathbf{c}}(r,k)$ of colored partitions in $\mathscr{P}^{\mathbf{c}}(r,k)$ with the additional structure of a specified diagonal with one of its vertices labeled.

Example

Let's look at $\mathscr{S}^{\mathbf{c}}(3,3)$ in terms of $\mathscr{P}^{\mathbf{c}}(3,3)\dots$



(Remember there are 5 different triangulations of a pentagon so $\left|\hat{p}(3,3)\right| = 5c_1^3$.)



On the one hand, since every partition by $k-1$ diagonals has $2(k-1)$ possibilities to label a vertex, we have

$$\left|\mathscr{S}^{\mathbf{c}}(r,k)\right| = 2(k-1)\left|\mathscr{P}^{\mathbf{c}}(r,k)\right| = 2(k-1)\hat{p}(r,k).$$

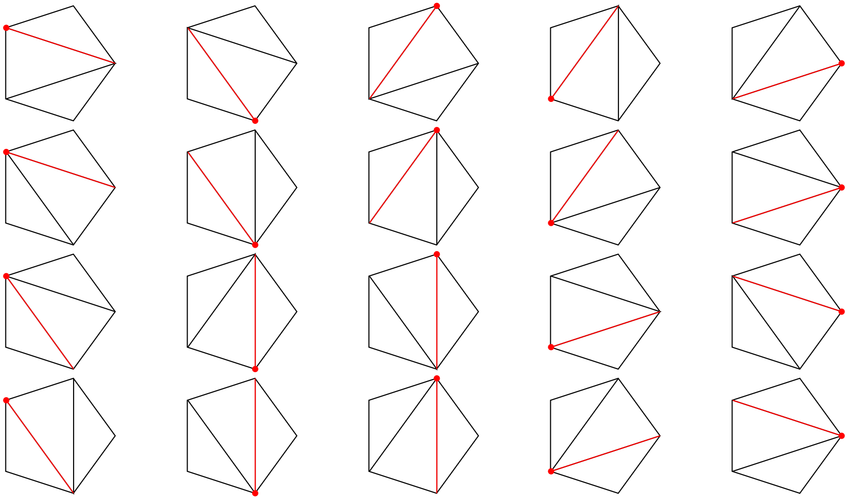
Example (continued)

$$\left|\mathscr{S}^{\mathbf{c}}(3,3)\right| = 4 \times 5c_1^3 = 20c_1^3$$

On the other hand, for any $(r+2)$-gon we can write $\mathscr{S}^{\mathbf{c}}(r,k)$ as a disjoint union

$$\mathscr{S}^{\mathbf{c}}(r,k) = \bigsqcup_{v} S_v(r,k)$$

over all vertices v of the polygon, where $S_v(r,k)$ is the subset of partitions having v as their labeled vertex.



Since $|S_v(r,k)|$ is independent of v, we have

$$|\mathscr{S}^{\mathbf{c}}(r,k)| = (r+2)|S_v(r,k)|.$$

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Let v be an arbitrary labeled vertex. Any partition in $S_v(r,k)$ has a distinguished diagonal (containing v) which dissects the $(r+2)$-gon into two polygons, one with $\ell - 1$ diagonals $(0 < \ell < k)$ and the other with $k - \ell$ diagonals.

Summing over all possible admissible values of ℓ and m (namely, $1 \leq \ell \leq k-1$ and $1 \leq m \leq n-1$) gives

$$|S_v(r,k)| = \sum_{\ell=1}^{k-1} \sum_{m=\ell}^{n-1} \hat{p}(n-m, k-\ell)\hat{p}(m,\ell).$$

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Finally, putting these parts together gives

$$(r+2) \sum_{\ell=1}^{k-1} \sum_{m=\ell}^{r-1} \hat{p}(r-m, k-\ell)\hat{p}(m,\ell) = 2(k-1)\hat{p}(r,k).$$

$$\sum_{\ell=1}^{k-1}\sum_{m=\ell}^{r-1}\hat{p}(r-m,k-\ell)\hat{p}(m,\ell)=\frac{2(k-1)}{r+2}\hat{p}(r,k).$$

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Bringing back our Bell convolution we can choose $a=1$ and $b=2$ to give

$$\hat{p}(r,k)=q(r,k)=\frac{\binom{r+k+1}{k}}{r+k+1}\cdot\frac{k!}{r!}B_{r,k}(1!c_1,2!c_2,\dots).$$

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Notice that summing over k is now counted by y_r.

Remark

We can choose our sequence of parameters, $\mathbf{c}$*, to get the partition of our choice. If we choose* $c_1=1$ *and* $c_k=0$ *for all* $k>1$ *we can reduce to Cayley's triangulation problem:* $\hat{p}(r,k)=\delta_{r,k}C_r$.

Example

$(a = 1,\ b = 2,\ c_1 = 1,\ c_2 = 1,\ c_j = 0,\ j > 2)$ For this case, y_r is counting the number of dissections of convex polygons with $r + 2$ sides into triangles and quadrilaterals by nonintersecting diagonals.

$$y_r = \sum_{k=1}^{r} \binom{r+k}{k-1} \frac{(k-1)!}{r!} B_{r,k}(1, 2, 0, 0, \dots) = \sum_{k=1}^{r} \binom{r+k}{k-1} \binom{k}{r-k} \frac{1}{k}.$$

y_r is the OEIS sequence A001002, $\{1,\ 3,\ 10,\ 38,\ 154,\ 654,\ 2871, \dots\}$.

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Example

$(a = 1,\ b = 2,\ c_1 = 0,\ c_j = 1,\ j > 1)$ For this case, y_r is counting the number of dissections of convex polygons with $r + 2$ sides so as to create no triangles.

$$y_r = \sum_{k=1}^{r} \binom{r+k}{k-1} \frac{(k-1)!}{r!} B_{r,k}(0, 2!, 3!, 4!, \dots) = \sum_{k=1}^{\lfloor r/2 \rfloor} \binom{r+k}{k-1} \binom{r-k-1}{k-1} \frac{1}{k}.$$

y_r is the OEIS sequence A046736, $\{0,\ 1,\ 1,\ 4,\ 8,\ 25,\ 64,\ 191,\ 540\ , \dots\}$.

Let T_n be the set of rooted trees with $n+1$ nodes such that the edges connecting a node with its k children may be colored in c_k different ways. Let t_n count the number of colored trees in T_n.

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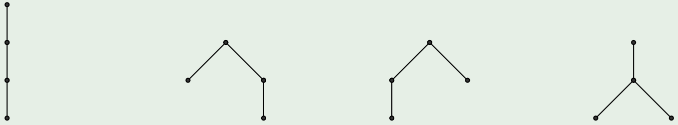
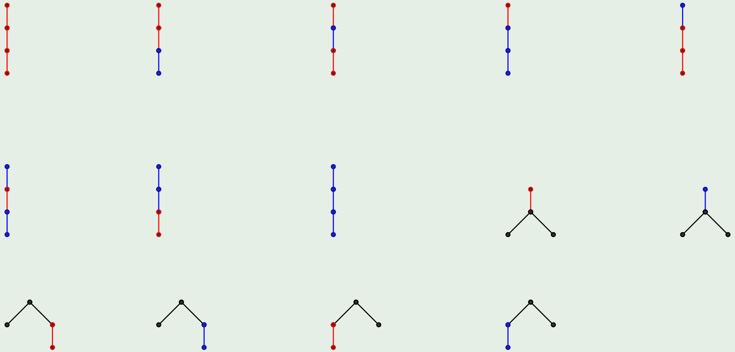
Example

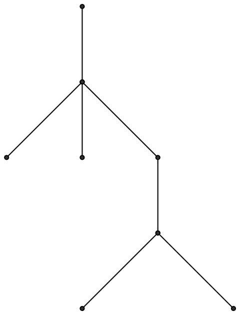
Setting $c_1 = 1$, $c_2 = 1$, and $c_k = 0$ for all $k > 2$, we retrieve our binary tree example and therefore $t_n = M_n$. For n=3, four trees:

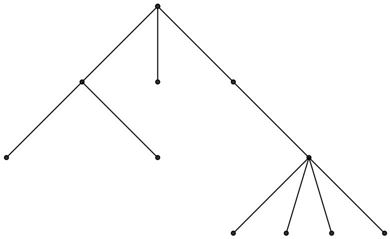


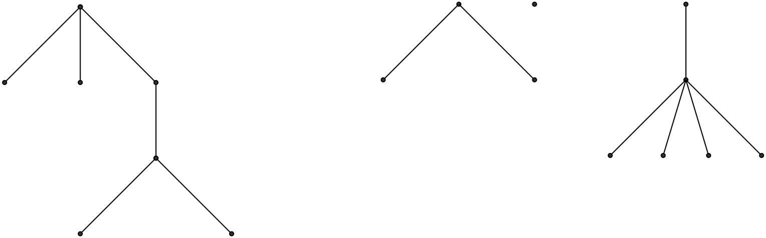
Example

Setting $c_1 = 2$, $c_2 = 1$, and $c_k = 0$ for all $k > 2$, we again allow at most two children but have two different colors (weights) available for single child edges. These are the (shifted) Catalan numbers with $t_n = C_{n+1}$. Below we have for $n = 3$, fourteen trees:









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Using our Bell convolution we can inductively see, for $r > 0$,

$$t_r = \sum_{d=1}^{r} c_d \sum_{m_1+\cdots+m_d=r-d} t_{m_1} \cdots t_{m_d}$$

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$$t_r = \sum_{d=1}^{r} c_d \sum_{m_1+\cdots+m_d=r-d} y_{m_1}\cdots y_{m_d} = \sum_{d=1}^{r} c_d\, d \sum_{i=0}^{r-d} \frac{\binom{r}{i}}{r} \frac{i!}{(r-d)!} B_{r-d,i}(\mathbf{c})$$

$$= c_r + \sum_{i=2}^{r} \binom{r-1}{i-2} \frac{(i-2)!}{(r-1)!} \left[\sum_{d=i-1}^{r-1} \binom{r-1}{d} (r-d)!\, c_{r-d} B_{d,i-1}(\mathbf{c}) \right]$$

$$= c_r + \sum_{i=2}^{r} \frac{1}{r+1} \binom{r+1}{i} \frac{i!}{r!} B_{r,i}(\mathbf{c}) = y_r.$$

□

Example

($c_{2j} = 1,\ c_{2j-1} = 0,\ j > 0$) For this case, y_r is counting the number of rooted trees consisting of nodes with only an even number of children.

$$y_{2r} = \sum_{k=1}^{2r} \binom{2r}{k-1} \frac{(k-1)!}{(2r)!} B_{2r,k}(0, 2!, 0, 4!, 0, 6!, \dots) = \frac{1}{2r+1}\binom{3r}{r}.$$

y_{2r} is the OEIS sequence A001764, { 1, 3, 12, 55, 273, 1428, 7752,... }.

Example

($c_1 = 0,\ c_2 = 1,\ c_3 = 1$, and $c_j = 0,\ j > 3$) For this case, y_r is counting the number of rooted trees consisting of nodes with either zero, two, or three children.

$$y_r = \sum_{k=1}^{r} \binom{r}{k-1} \frac{(k-1)!}{r!} B_{r,k}(0, 2!, 3!, 0, 0, \dots) = \sum_{k=1}^{\lfloor r/2 \rfloor} \binom{r}{k-1} \binom{k}{r-2k} \frac{1}{k}.$$

y_r is the OEIS sequence A001005, {0, 1, 1, 2, 5, 8, 21, 42, 96, 222, 495,... }.

For any α and β, and any sequence (c_j),

$$q(r,k) = \frac{\binom{\alpha r+\beta k+1}{k}}{\alpha r+\beta k+1} \cdot \frac{k!}{r!} B_{r,k}(1!c_1, 2!c_2, \dots)$$

counts the number of:

(a) colored partitions of polygons with $\alpha r + (\beta - 1)k + 2$ sides into k "blocks" with allowable blocks $(\alpha j + \beta + 1)$-gons, c_j colors.

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counts the number of:

(a) colored partitions of polygons with $\alpha r + (\beta - 1)k + 2$ sides into k "blocks" with allowable blocks $(\alpha j + \beta + 1)$-gons, c_j colors.

(b) weighted trees with $(\alpha + \beta)r + 1$ nodes made from k "blocks" with allowable blocks as subtrees with $(\alpha + \beta)j + 1$ nodes, $\alpha j + \beta$ leaves and weight c_j.

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(c) colored Dyck paths of length $2(\alpha + \beta)r$ having k peaks made from blocks of the form $U^{(\alpha+\beta)j} D^{\beta(j-1)+1}$ w/ c_j colors, & D

Question

How many ways can we dissect a convex dodecagon (12 sided polygon) by non-intersecting diagonals into only pentagons and hexagons?

Question

How many 11-node rooted trees are there if each non-leaf node can only have either 1 child or 4 children and there are exactly 4 nodes with children? What if exactly 7 nodes have children?

Question

How many Dyck paths of semi-length 10 have exactly 3 peaks but no maximal ascent sequences of odd length?

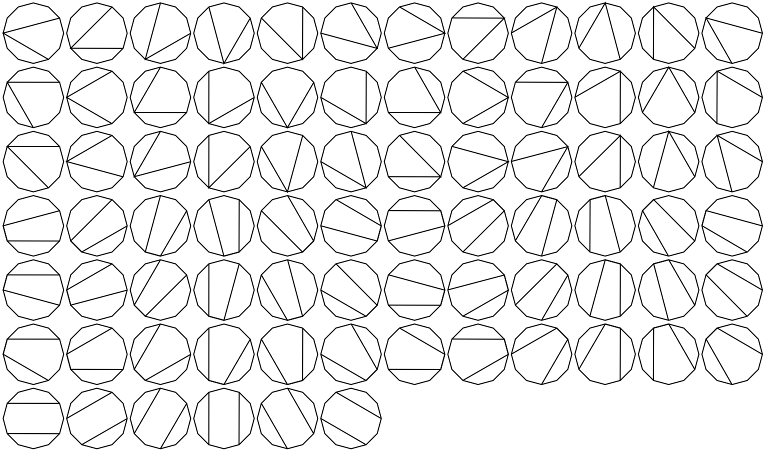
Using $\alpha = 1$, $\beta = 1$, $c_3 = 1$, $c_4 = 1$, and $c_j = 0$ for all other j,

$$\begin{aligned} y_r &= \sum_{i=0}^{r} \frac{\binom{r+i+1}{i}}{r+i+1} \cdot \frac{i!}{r!} B_{r,i}(0,0,3!,4!,0,\dots) \\ &= \sum_{i=0}^{r} \frac{\binom{r+i+1}{i}}{r+i+1} \cdot \frac{i!}{(r-2i)!} B_{r-2i,i}(1!,2!,0,0,\dots) \\ &= \frac{1}{r+1} \sum_{i=0}^{\lfloor r/3 \rfloor} \binom{r+i}{i} \binom{i}{r-3i}, \end{aligned}$$

so $y_{10} = 78$.

{0,0,1,1,0,4,9,5,22,78,91,175,680,1224,1938,6270,14630,24794,63756,166980,...}

{This sequence is currently not in the OEIS.}



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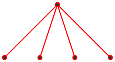
Using $\alpha = 1$, $\beta = 0$, (weighted trees with $r + 1$ nodes), $c_1 = 1$, $c_4 = 1$, and $c_j = 0$ for all other j, our building blocks become:

c_1

c_4



Example trees w/4 blocks

$r = 10, \ k = 4$

Using $\alpha = 1$, $\beta = 0$, $c_1 = 1$, $c_4 = 1$, and $c_j = 0$ for all other j,

$$q(r,k) = \frac{\binom{r+1}{k}}{r+1} \cdot \frac{k!}{r!} B_{r,k}(1!, 0, 0, 4!, 0, \dots)$$
$$= \begin{cases} \frac{1}{r+1}\binom{r+1}{k}\binom{k}{\frac{1}{3}(r-k)}, & \frac{r-k}{3} \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$$

and therefore $q(10,4) = 180$ and $q(10,7) = 210$.

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Dyck path example: $\alpha = 2$, $\beta = 0$, $r = 5$, $k = 3$

$$(U^4D)D(U^4D)D^2(U^2D)D^4$$

$$(c_2) \quad D \quad (c_2) \quad D^2 \quad (c_1) \quad D^4$$

Using $\alpha = 2$, $\beta = 0$, and $c_j = 1$ for all j,

$$q(r,k) = \frac{\binom{2r+1}{k}}{2r+1} \cdot \frac{k!}{r!} B_{r,k}(1!, 2!, 3!, 4!, 5!, \dots)$$
$$= \frac{1}{r}\binom{r}{k}\binom{2r}{k-1}$$

so $q(10,3) = 90$.

Using $\alpha = 2$, $\beta = 0$, and $c_j = 1$ for all j,

$$q(r,k) = \frac{\binom{2r+1}{k}}{2r+1} \cdot \frac{k!}{r!} B_{r,k}(1!, 2!, 3!, 4!, 5!, \dots) = \frac{1}{r}\binom{r}{k}\binom{2r}{k-1}$$

so $q(10, 3) = 90$.

Note: Letting $\alpha = 1$, $\beta = 0$, and $c_j = 1$ for all j recovers the Narayana numbers:

$$q(r,k) = \frac{\binom{r+1}{k}}{r+1} \cdot \frac{k!}{r!} B_{r,k}(1!, 2!, 3!, 4!, 5!, \dots) = \binom{r}{k-1}\binom{r}{k}\frac{1}{r} = N(r,k).$$

Thank You

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Some of the material presented here can be found in:

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3. J. Riordan, *An Introduction to Combinatorial Analysis*, Wiley, New York, 1958.
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6. C.A. Charalambides, *Enumerative Combinatorics*, Chapman and Hall/CRC, Boca Raton, 2002.
7. D. Birmajer, J. Gil and M. Weiner, *Some convolution identities and an inverse relation involving partial Bell polynomials*, Electron. J. Combin. **19** (2012), no. 4, Paper 34, 14 pp.
8. D. Birmajer, J. Gil and M. Weiner, *Convolutions of tribonacci, Fuss-Catalan, and Motzkin sequences*, Fibonacci Quart. accepted December 2014.
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Bell Convolution Formula

Corollary (BGW 2012[4])

Let $\alpha(\ell, m)$ be a polynomial in ℓ and m of degree at most one. For any sequence $x = (x_1, x_2, \dots)$ and any $\tau \in \mathbb{C}$, we have

$$\sum_{\ell=0}^{k} \sum_{m=\ell}^{n} \frac{\tau \binom{\alpha(\ell,m)}{k-\ell} \binom{\tau-\alpha(\ell,m)}{\ell} \binom{n}{m}}{\alpha(\ell,m)\big(\tau-\alpha(\ell,m)\big)\binom{k}{\ell}} B_{m,\ell}(x) B_{n-m,k-\ell}(x)$$

$$= \frac{\tau-\alpha(0,0)+\alpha(k,n)}{\alpha(k,n)(\tau-\alpha(0,0))} \binom{\tau}{k} B_{n,k}(x).$$

where $B_{i,j}(x) = B_{i,j}(x_1, x_2, \dots)$ for any fixed sequence $x_1, x_2, \dots$.

[4] Birmajer, Gil and Weiner, *Some convolution identities and an inverse relation involving partial Bell polynomials*, Electron. J. Combin. **19** (2012), no. 4, Paper 34, 14 pp.