When is a double central extension *universal*?

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joint work with George Peschke

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Introduction

- The concept of *universality* of central extensions extends from the context of (co)homology of groups to semi-abelian categories under mild additional assumptions.
- Universal central extensions occur when H_1 vanishes, so that in this case the interplay between H_2 (Hopf formula) and H^2 (central extensions) becomes particularly nice.
- For $n > 1$, an interpretation of H_{n+1} through higher Hopf formulae and H^{n+1} via n-fold central extensions has recently become available. Whence the question:

What does it mean for an *n*-fold central extension to be *universal*?

The aim of my talk is to explain

1. why the naive approach fails;
2. how to correct this, so that a general existence result is obtained;
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Universal central extensions of groups, I

An **extension** under A and over X is a short exact sequence

$$0 \longrightarrow A \rightarrowtail E \xrightarrow{f} X \longrightarrow 0.$$

It is **central** if and only of $[A, E] = 0$: all $aea^{-1}e^{-1}$ vanish, $a \in A$, $e \in E$. Then, in particular, A is an abelian group.

Theorem
Considering A as a trivial X-module, we get $\mathrm{H}^2(X, A) \cong \mathrm{Centr}^1(X, A)$, the group of equivalence classes of central extensions under A and over X.

- By the Short Five Lemma, equivalence class = isomorphism class.

The theorem remains true [Gran & VdL, 2008] in any semi-abelian category [Janelidze, Márki & Tholen, 2002] with enough projectives; centrality is now defined via categorical Galois theory [Janelidze & Kelly, 1994].

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Universal central extensions of groups, II

X is **perfect** when $[X,X]=X$, that is, $X/[X,X]=\mathrm{ab}(X)=\mathrm{H}_1(X)=0$.

A central extension $u\colon U\to X$ is **universal** iff it is initial in $\mathrm{CExt}_X(\mathrm{Gp})$:

For perfect groups, computing H_2 is easy

- If $u\colon U\to X$ is universal then X and U are perfect, while $\mathrm{Ker}(u)\cong \mathrm{H}_2(X)$; conversely,
- any perfect X admits a universal central extension, unique up to iso.

Construction for X perfect

1. Take a projective presentation $R\to F\to X$;
2. centralise to a weakly universal central extension $\frac{R}{[R,F]}\to\frac{F}{[R,F]}\to X$;
3. take commutators: $\frac{R\wedge[F,F]}{[R,F]}\to\left[\frac{F}{[R,F]},\frac{F}{[R,F]}\right]\to[X,X]$.

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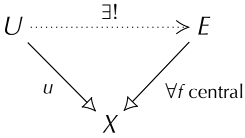

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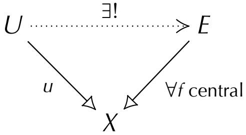

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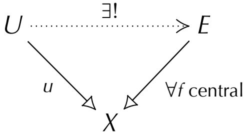

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Universal central extensions of groups, III

Recognition theorem

For any perfect group U, the following are equivalent:

1. every central extension $u\colon U \to X$ is universal;
2. all universal central extensions of U split;
3. U is *superperfect*: $H_1(U) = H_2(U) = 0$.

By [Casas & VdL, 2014] combined with [Gray & VdL, 2014], this remains true in semi-abelian categories with enough projectives which are *peri-abelian* [Bourn, 2010].

- Algebraic coherence $\Rightarrow$ peri-abelianness [Cigoli, Gray & VdL, 2015]
- Hence we find as examples: associative algebras; Lie, Leibniz, Poisson algebras; n-nilpotent or n-solvable groups, rings, algebras; torsion-free groups, reduced rings; crossed modules of such; n-cat-groups; compact Hausdorff models of such.

Universal *double* central extensions for *super*perfect objects?

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Higher central extensions and cohomology

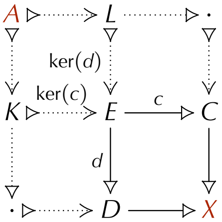
Theorem [Rodelo & VdL, 2011, 2012]

In a semi-abelian category $\mathbb{X}$ with enough projectives satisfying (SH), for any $X \in \mathbb{X}$, $A \in \mathrm{Ab}(\mathbb{X})$ and $n \geqslant 1$,

$$\mathrm{H}^{n+1}(X, A) \cong \mathrm{Centr}^{n}(X, A)$$

where the group on the right consists of equivalence classes of *n-fold central extensions* in the sense of [Everaert, Gran & VdL, 2008] and [Janelidze, 1991] over X and under A.

- We focus on the case $n = 2$: double central extensions
- Extension = regular pushout
  $\Leftrightarrow 3 \times 3$ diagram
  Classes determined by zigzags
- $A = K \wedge L$
  Central $\Leftrightarrow [K, L] = [A, E] = 0$



Naive approach: when is a double central extension *initial* over X?

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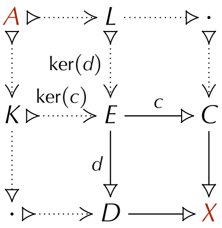

Naive approach: when is a double central extension *initial* over X?

If X admits an initial double central extension then $X = 0$

- Suppose the double central extension on the left is initial.

- Consider $B \in \mathrm{Ab}(\mathbb{X})$, $X, Y \in \mathbb{X}$ and the 3×3 diagram on the right. The square is a double central extension, because $[B, Y \times B] = [B, X \times Y \times B] = 0$ since B is abelian.
- Hence there exists a unique arrow making the diagram commute. So for any chosen object $Y \in \mathbb{X}$, there is a unique $y\colon C \to Y$.
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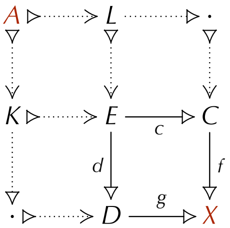

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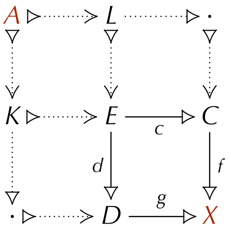

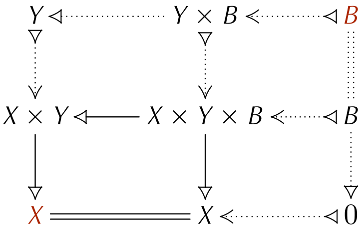

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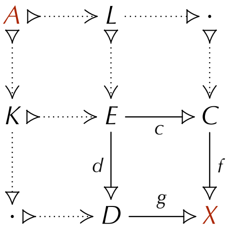

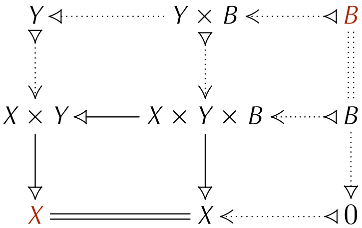

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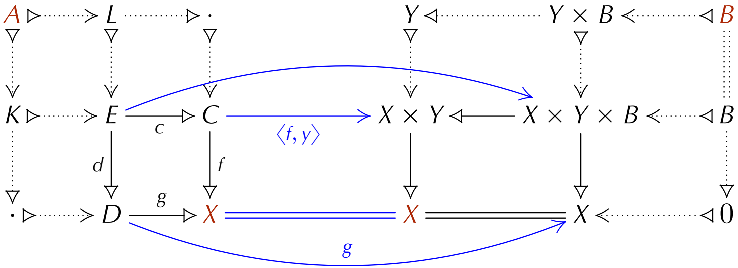

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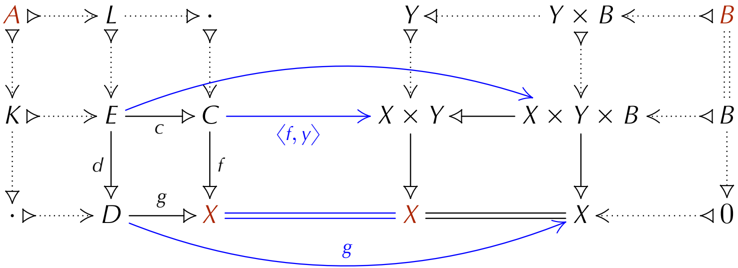

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Universal one-fold central extensions revisited

Proposition

For X perfect, a central extension $u \in \overline{u} \in \mathrm{Centr}^1(X, H)$ is universal iff

$$\mu^{\overline{u}} \colon \mathrm{Hom}(H, -) \to \mathrm{Centr}^1(X, -) \colon \mathrm{Ab}(\mathbb{X}) \to \mathrm{Ab}$$

$$(h \colon H \to A) \mapsto \mathrm{Centr}^1(X, h)(\overline{u})$$

determines a natural isomorphism.

This means that $(H, \overline{u})$ is a *universal element* of $\mathrm{Centr}^1(X, -)$.

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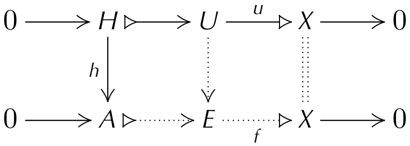

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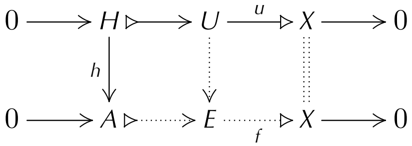

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Universal n-fold central extensions, $n \geqslant 1$

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In other words, $(H, \overline{U})$ is a universal element of $\mathrm{Centr}^n(X, -)$.

Theorem [Peschke & VdL, 2015]

In a semi-abelian variety satisfying (SH), for any X such that

$$\mathrm{H}_1(X) = \mathrm{H}_2(X) = \cdots = \mathrm{H}_n(X) = 0$$

there exists a universal n-fold central extension, unique up to equivalence, over X and under $H = \mathrm{H}_{n+1}(X)$.

How to construct a universal double central extension?

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determines a natural isomorphism.

In other words, $(H,\overline{U})$ is a universal element of $\mathrm{Centr}^n(X,-)$.

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In a semi-abelian variety satisfying (SH), for any X such that

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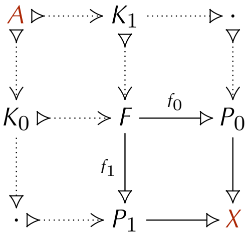

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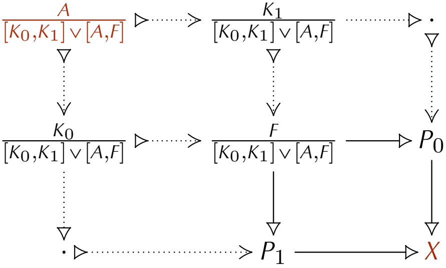

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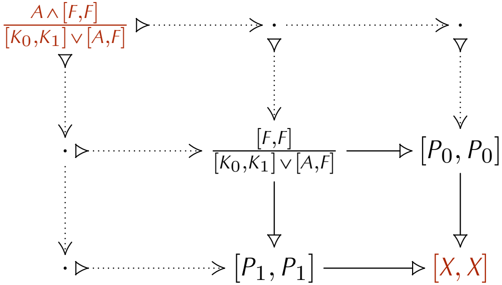

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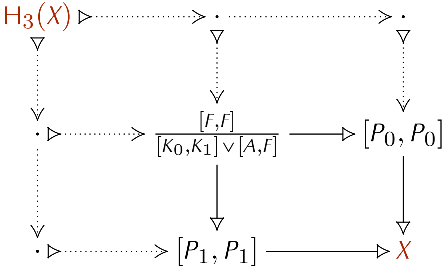

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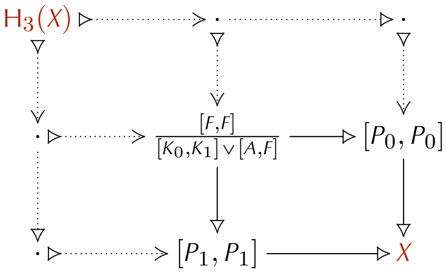

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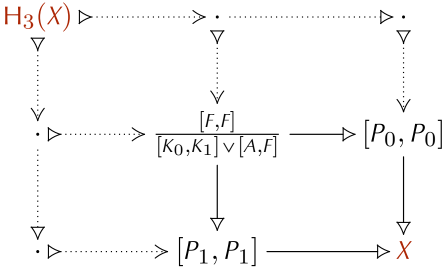

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- This universal higher central extension may be obtained via a construction which mimics the case $n = 1$.
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