Coulomb branch of 3d $\mathcal{N} = 4$ theories and the moduli space of instantons

Stefano Cremonesi

King's College London

AMS / EMS / SPM International Meeting
Porto, June 12, 2015

Based on:

- SC, A. Hanany, A. Zaffaroni [arXiv:1309.2657]
- SC, G. Ferlito, A. Hanany, N. Mekareeya [arXiv:1408.6835]

What we study:

- The Coulomb branch of the moduli space of infrared fixed points of 3d $\mathcal{N} = 4$ gauge theories.

What we compute:

- The Hilbert series: generating function counting chiral operators.

Why we do it:

- 3d mirror symmetry (tests/predictions)
- HyperKähler cones without HK quotient
- Instanton moduli spaces without ADHM **(today)**

Dressed monopole operators and Hilbert series of the Coulomb branch

Moduli space of vacua of 3d $\mathcal{N} = 4$ gauge theories

HIGGS branch $\mathcal{M}_H$

- Hyper-plets H_i, $i = 1, \ldots, n$
- $SU(2)_H$ R-symmetry
- HyperKähler quotient

$$\{\vec{\mu}(H_i) = \vec{\xi}\}/G$$

e.g. ADHM construction

- $dim_{\mathbb{H}} \mathcal{M}_H = n - d$
- Classically exact

COULOMB branch $\mathcal{M}_C$

- Vector-plets V, gauge group $\mathcal{G}$
- $SU(2)_C$ R-symmetry
- HyperKähler:

$$(\vec{\sigma}_a, \varphi_a) \,, \quad d\varphi_a = *F_a$$

at generic pts ($\mathcal{G} \to U(1)^{\mathrm{rk}(\mathcal{G})}$)

- $dim_{\mathbb{H}} \mathcal{M}_C = \mathrm{rk}(\mathcal{G})$
- Quantum corrections

$V_m(x) \sim e^{-\sum_a m_a(\sigma_a + i\varphi_a)/g^2_{a,eff}(\sigma)}(x)$:

- well-defined at generic points on the Coulomb branch $\mathcal{M}_C$
- introduce magnetic charges m_a at insertion point x: monopole operators

Monopole operators

['t Hooft 1978] [Borokhov, Kapustin, Wu 2002] [Kapustin 2005]

- **Local disorder operator $\mathcal{O}(x)$**: prescribed singularity in the Euclidean path integral at insertion point x.

- **Monopole operator $V_m(x)$**: Dirac monopole singularity $U(1) \hookrightarrow \mathcal{G}$.

Monopole operator $V_m(0)$

$$A_\pm = \frac{m}{2}(\pm 1 - \cos\theta)d\varphi + \mathcal{O}(r^0) \ , \qquad r \to 0$$

Magnetic charge $m \in \mathfrak{t} = \mathrm{Cartan}(\mathfrak{g})$, modulo Weyl group $\mathcal{W}_\mathcal{G}$.

Generalised Dirac quantisation [Englert, Windey 1976]

$$\exp(2\pi i m) = \mathbb{I}_\mathcal{G}$$

$\Rightarrow m \in \Gamma_{\mathcal{G}^\vee}$ weight lattice of $\mathcal{G}^\vee$ (dual group of $\mathcal{G}$). [Goddard, Nuyts, Olive 1977]

Supersymmetric 't Hooft monopole operators

They parameterise the Coulomb branch $\mathcal{M}_C$ of a 3d susy gauge theory.

We study 3d $\mathcal{N} = 4$, but work in $\mathcal{N} = 2$ formalism (fixed complex structure):

$\mathcal{N} = 4$ Vector Multiplet

- $\mathcal{N} = 2$ Vector Multiplet: A_μ, $\sigma \in \mathbb{R}$, fermions, aux.
- $\mathcal{N} = 2$ Chiral Multiplet: $\phi \in \mathbb{C}$, fermions, aux.

$\mathcal{N} = 2$ supersymmetric monopole operator (Chiral Multiplet)

$$A_\pm = \frac{m}{2}(\pm 1 - \cos\theta)d\varphi + \mathcal{O}(1) \qquad r \to 0$$

$$\sigma = \frac{m}{2r} + \mathcal{O}(1) \qquad r \to 0$$

[Borokhov, Kapustin, Wu 2002] [Borokhov 2003]

$\mathcal{N} = 2$ BPS equation: $(d - iA)\sigma = - * F$.

Charges of bare monopole operators

- **Classical**: Topological symmetry $G_J = Z(\mathcal{G}^\vee)$

Topological charge of monopole operator $V_m(x)$

$$e.g. \text{ for } \mathcal{G} = U(N): \qquad J(m) = \mathrm{Tr}(m)$$

- **Quantum**: R-symmetry $U(1)_{R_C} \subset SU(2)_C \qquad \Delta = R_C$

R-charge of bare monopole operator $V_m(x)$ [Bashkirov, Kapustin 2010]

$$\Delta(m) = -\sum_{\alpha\in\Delta_+} |\alpha(m)| + \frac{1}{2}\sum_{i=1}^{n}\sum_{\rho_i\in\mathcal{R}_i} |\rho_i(m)|$$

Focus on *good/ugly* gauge theories [Gaiotto, Witten 2008]: $\Delta(m) \geq \frac{1}{2}\ \forall m$.

Δ is the scaling dimension of BPS operators in the IR SCFT.

Dressed monopole operators

[SC, Hanany, Zaffaroni 2013]

We have only used the $\mathcal{N} = 2$ Vector Multiplet. Can we turn on ϕ?

Adjoint breaking by magnetic flux m

$$\mathcal{G} \quad \xrightarrow{\ m\ } \quad H_m$$

$$\text{s.t. } [m, H_m] = 0$$

BPS moduli: constant $\phi = \phi_m \in \mathfrak{h}_m$

$\phi_m = 0$: "Bare" monopole operators

$\phi_m \neq 0$: "Dressed" monopole operators

These monopole operators preserve the SUSY of an $\mathcal{N} = 2$ chiral multiplet.

Hilbert series of the Coulomb branch

[SC, Hanany, Zaffaroni 2013]

Generating function that counts gauge invariant chiral dressed monopole op's (holomorphic functions on $\mathcal{M}_C$).

$$H_{\mathcal{M}_C}(t,z) \equiv \mathrm{Tr}_{\mathcal{H}_C}(z^J t^{2\Delta}) = \sum_{m \in \Gamma_{\mathcal{G}^\vee}/\mathcal{W}_{\mathcal{G}}} z^{J(m)} t^{2\Delta(m)} P_{\mathcal{G}}(t^2; m)$$

- $\mathcal{H}_C$: ring of chiral operators on $\mathcal{M}_C$
- $\sum_m$: sum over magnetic charges
- $z^{J(m)}$: topological charge in $Z(\mathcal{G}^\vee)$
- $t^{2\Delta(m)}$: R_C-charge / dimension of bare monopole operator V_m
- $P_{\mathcal{G}}(t^2; m)$: dressing by adjoint $\phi_m \in \mathfrak{h}_m$

$$P_{\mathcal{G}}(t^2; m) = \prod_{i=1}^{\mathrm{rk}(\mathcal{G})} \frac{1}{1 - t^{2d_i(m)}}$$

$d_i(m)$: degrees of independent Casimirs of H_m.

Limit of superconformal index: $\mathcal{I}^C(t,z) = \mathrm{Tr}_{\mathcal{H}_C}(-1)^F z^J t^{2(R_C - R_H)}$ [Razamat, Willett 2014]

Example: $\mathcal{G} = SU(2)$ with $n > 2$ flavours

$$H_{\mathcal{M}_C}(t) = \frac{1}{1-t^4} + \frac{1}{1-t^2}\sum_{m=1}^{\infty} t^{2(n-2)m} = \frac{1-t^{4(n-1)}}{(1-t^4)(1-t^{2(n-2)})(1-t^{2(n-1)})}$$
$$= \mathrm{PE}[t^4 + t^{2(n-2)} + t^{2(n-1)} - t^{4(n-1)}]$$

Term	Interpretation	
t^4	$w = \frac{1}{2}\mathrm{Tr}(\phi^2) = \varphi^2$	Casimir invariant
$t^{2(n-2)}$	$u = \frac{1}{2}(V_1 + V_{-1})$	bare monopole operator
$t^{2(n-1)}$	$v = \frac{i}{2}(V_1 - V_{-1})\varphi$	dressed monopole operator
$-t^{4(n-1)}$	$v^2 + u^2 w = w^{n-1}$	relation

showing that $\mathcal{M}_C = \mathbb{C}^2/\Gamma_{D_n}$. [Intriligator, Seiberg 1996] [Borokhov 2003]

Hilbert series for moduli spaces of instantons

Moduli spaces of instantons as Coulomb branches

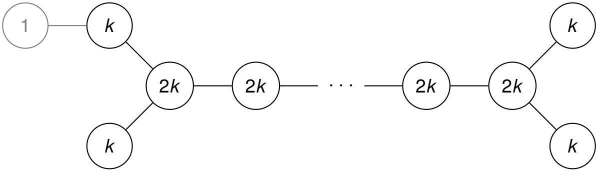
k D2 + 1 D6 branes transverse to ADE singularity $\mathbb{C}^2/\Gamma_G$.

$\rightarrow k$ M2 branes probing $\mathbb{C}^2/\Gamma_G \times \mathbb{C}^2$.

Quiver: affine ADE Dynkin diagram + extra node $/U(1)$, *e.g.* for D

[Intriligator, Seiberg 1996] [de Boer, Hori, Ooguri, Oz 1996]

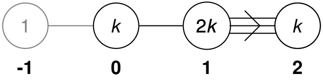
[Kronheimer Nakajima 1990]



Coulomb branch $\mathcal{M}_C = \mathcal{M}_{G,k}$ moduli space of k G-instantons on $\mathbb{C}^2$.

Hilbert series formula for Coulomb branch of over-extended Dynkin “quivers”.

E.g. for $G = G_2$

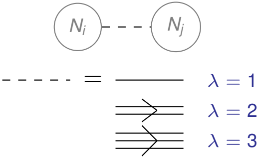


- **Nodes**: $U(N_i)$ gauge groups, with ranks

$$N_{-1} = 1 \,, \qquad N_i = k a_i^\vee \,, \qquad i = 0, \ldots, r \equiv \mathrm{rk}(G) \,.$$

$$\Delta_{\text{node}}(m^{(i)}) = -\sum_{1 \le a < b \le N_i} \left| m_a^{(i)} - m_b^{(i)} \right|$$

- **Laces**:



$$\Delta_{\text{lace}}(m^{(i)}, m^{(j)}) = \frac{1}{2} \sum_{a=1}^{N_i} \sum_{b=1}^{N_j} \left| \lambda m_a^{(i)} - m_b^{(j)} \right|$$

- **Remove decoupled $U(1)$**: shift symmetry $(c \in \mathbb{Z})$

$$m^{(-1)} \to m^{(-1)} + c\,, \qquad m^{(i)} \to m^{(i)} + c\,\frac{a_i}{a_i^\vee}\,\mathbb{I}_{ka_i^\vee} \quad (i = 0, \ldots, r)\,.$$

$\Rightarrow$ Dimension: $\dim_{\mathbb{H}} \mathcal{M}_C = k \sum_{i=0}^{r} a_i^\vee = kh^\vee = \dim_{\mathbb{H}} \mathcal{M}_{G,k}$.

- **Topological symmetry $G_J = U(1)^{2+r}/U(1)$**:

fugacities $z_{-1}, z_0, \ldots, z_r$ subject to $z_{-1}\left(\prod_{i=0}^{r} z_i^{a_i}\right)^k = 1$.

$$G_J = U(1)^{1+r} \text{ enhances to } \begin{cases} SU(2)_x \times G_u & k = 1 \\ SU(2)_x \times SU(2)_x \times G_u & k > 1 \end{cases}$$

due to monopole operators.

$$x = \prod_{i=0}^{r} z_i^{a_i}$$

$$u_i = z_i\,, \quad i = 1, \ldots, r$$

$$z_{-1} = x^{-k}$$

$$z_0 = x \prod_{i=1}^{r} u_i^{-a_i}$$

$$z_i = u_i\,, \quad i = 1, \ldots, r$$

- **Centre of mass** $\mathbb{C}^2$ generated by monopole operators (free fields)

$$H_{\mathcal{M}_{G,k}}(t,x,u) = \frac{1}{(1-tx)(1-tx^{-1})} H_{\widetilde{\mathcal{M}}_{G,k}}(t,x,u)$$

$$\downarrow$$

$$H_{\mathcal{M}_{G,k}}(t,x_1,x_2,u) = \frac{1}{(1-tx_1)(1-tx_1^{-1})} H_{\widetilde{\mathcal{M}}_{G,k}}(t,x_2,u)$$

- **Conjecture for low-lying generators of** $\mathcal{M}_{G,k}$:

$$H_{\mathcal{M}_{G,k}}(t,x,u) = \mathrm{PE}\left[\sum_{p=1}^{k}[p;0]_{x,u}t^p + \sum_{q=0}^{k-1}[q;\mathrm{Adj}]_{x,u}t^{q+2} - \mathcal{O}(t^{k+2})\right]$$

Generators are dressed monopole operators: balanced affine quiver makes it easy to identify them explicitly.

Outlook

Main result:

A simple formula for the Hilbert Series of the Coulomb branch of 3d $\mathcal{N} = 4$ SCFTs arising from good/ugly gauge theories:

- Characterises $\mathcal{M}_C$ algebraically by its holomorphic functions.
  see also [Nakajima 2015] [Bullimore, Dimofte, Gaiotto 2015]
- Predicts quantum numbers of generators and relations in the chiral ring.
- Reduces computation of chiral correlators to the generators only.
- Recently extended to 3d $\mathcal{N} = 2$. [SC 2015] [Hanany, Hwang, Kim, Park, Seong '15]

Application to instanton moduli spaces $\mathcal{M}_{G,k}$:

- Universal formalism for any G and k: group theory is transparent.
- Derive formula for non-simply laced G by folding?
- Prove conjecture on generators? Understand relations?