A simple model for
Coarsening in Infinite Particle Systems

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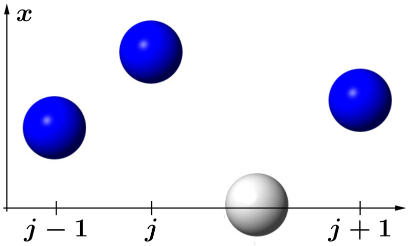
A simple model

Particles $x_j > 0$, $j \in \mathbb{Z}$ governed by

$$\dot{x}_j = x_{j+1}^{-\beta} - 2x_j^{-\beta} + x_{j-1}^{-\beta} = \text{“}\Delta(x_j^{-\beta})\text{”}$$

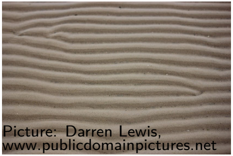
where vanished particles $x_j = 0$ are removed and the remaining ones are remeshed.



A simple model

Applications

Equations of type $\dot{x}_j =$ "$\Delta g(x_j)$" appear e. g. in



- convective Cahn-Hilliard (Watson, Otto, Rubinstein, Davis 2003),
- models for sand ripples (Hellén, Krug 2002).

Mathematical Interest

- toy model for neighbourhood interactions,
- rigorous statistical description (coarsening, self-similarity),
- well-posedness for infinite systems.

Existence of solutions

Screening in spatially homogeneous systems

Large particles

$$x_{j_k} \geq d, \qquad j_{k+1} - j_k \leq L,$$

where $d, L = \mathrm{const} > 0$,

- remain large, say $\geq d/2$, up to some time T_d^* due to $\dot{x}_{j_k} \geq -2x_{j_k}^{-\beta}$,
- trap mass $M_{k,l}(t) = \sum_{j=k}^{l} x_j(t)$ since

$$\begin{aligned} M_{j_k,j_l}(t) &\geq M_{j_k,j_l}(0) \qquad - C_d t, \\ M_{j_k+1,j_l-1}(t) &\leq M_{j_k+1,j_l-1}(0) + C_d t. \end{aligned}$$

Existence of solutions

Screening in spatially homogenous systems

➔ Homogeneity

$$\frac{1}{L} M_{j_k, j_{k+1}-1}(0) \geq d \qquad \text{for all } k \in \mathbb{Z}$$

is preserved up to time T_d^* in the sense

$$\frac{1}{qL} M_{j_k, j_{k+q}-1}(t) \geq (1 - 1/q) d \qquad \text{for all } k, q \in \mathbb{Z}.$$

Existence of solutions

Global-in-time existence follows from

- truncation $x^n = (x^n_{-n}, \ldots, x^n_n)$,
- uniform estimates as $n \to \infty$ (depending on L, d) and the Arzelà-Ascoli Theorem,
- iteration $L \to qL$, $d \to (1 - 1/q)d$, $T^*_d \to T^*_d + T^*_{(1-1/q)d}$,

Solution means

- x_j continuous and $x_j^{-\beta}\chi_{\{x_j>0\}}$ locally integrable,
- integral equation

$$x_j(t_2) - x_j(t_1) = \int_{t_1}^{t_2} \Delta_{\sigma(j,x(s))}\big(x_j(s)^{-\beta}\big)\,\mathrm{d}s$$

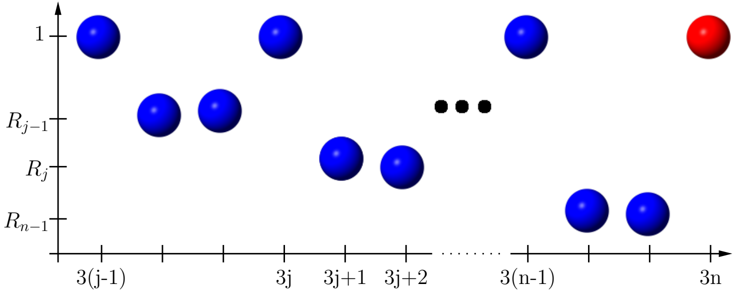
where Δ_σ incorporates neighbourhoods and vanishing.

Non-uniqueness

Initial data

$$x_{3j}(0) = 1, \qquad x_{3j+1}(0) \approx x_{3j+2}(0) \approx R_j$$

for $j = 1, 2, \ldots, n-1$ where (R_j) decreases exponentially



and ...

Non-uniqueness

Perturbation

either $x_{3n}(0) = 1$ ➔ truncated solution $\overline{x}^n$

or $x_{3n}(0) = 1 + \varepsilon_n^n$ ➔ truncated solution x^n

is propagated through the solution via

$$|\overline{x}^n_{3(n-1)}(T^n_{n-1}) - x^n_{3(n-1)}(T^n_{n-1})| = \varepsilon^n_{n-1} > \varepsilon^n_n$$
$$|\overline{x}^n_{3(n-2)}(T^n_{n-2}) - x^n_{3(n-2)}(T^n_{n-2})| = \varepsilon^n_{n-2} > \varepsilon^n_{n-1}$$
$$\vdots$$

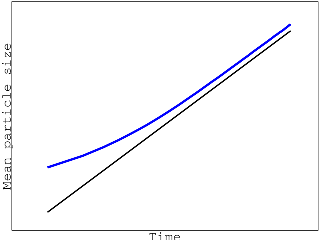
and as $n \to \infty$ with $\varepsilon_n^n \to 0$ yields two different solutions $\overline{x}$ and x.

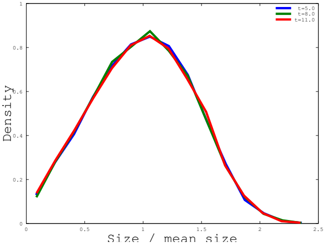
Coarsening dynamics

Features of coarsening model

- particles disappear in finite time,
- mass $\sum x_j$ is (formally or locally) conserved

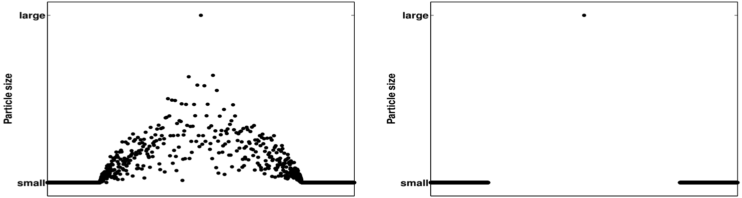
Generic behaviour in finite systems: $\langle x \rangle \sim t^{\frac{1}{\beta+1}}$





Coarsening dynamics in the infinite system

- $\langle x \rangle \lesssim T^{\frac{1}{\beta+1}}$ where T is the time to transfer a certain amount of average mass from vanishing to surviving particles;
- interesting non-generic behaviour



- explicit solutions with nearly generic coarsening rate by "organized coarsening structures"