Comparing skew Schur functions: a quasisymmetric perspective

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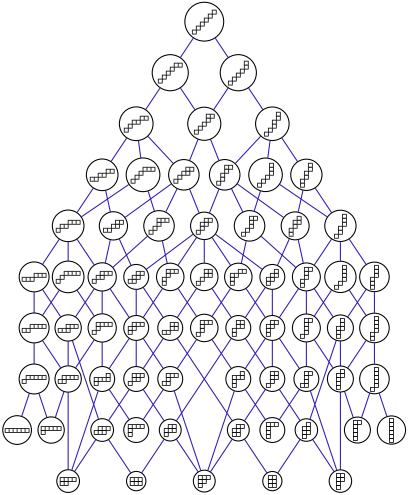
Slides and paper available from
`www.facstaff.bucknell.edu/pm040/`

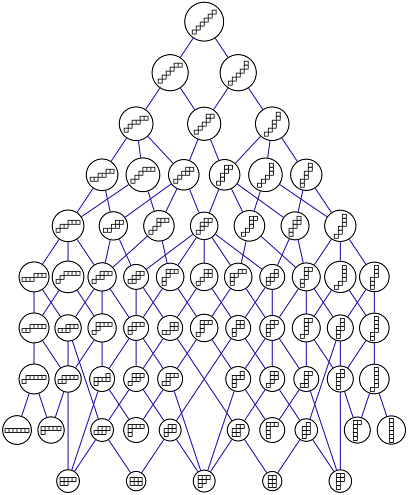
Outline

- The background story: the equality question
- Conditions for Schur-positivity
- Quasisymmetric insights and the main conjecture

Preview



F-support containment

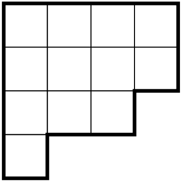


Dual of row overlap dominance

Cauchy, 1815

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$

- Young diagram.
  Example:
  $\lambda \quad = (4, 4, 3, 1)$



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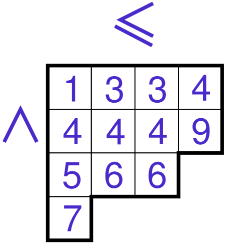

Schur functions

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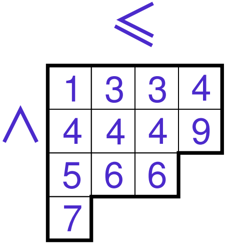


The Schur function s_λ in the variables $x = (x_1, x_2, \ldots)$ is then defined by

$$s_\lambda = \sum_{\text{SSYT } T} x_1^{\#1\text{'s in } T} x_2^{\#2\text{'s in } T} \cdots .$$

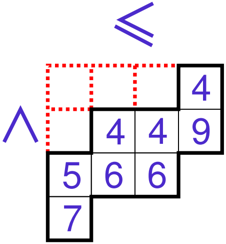
Example.

$s_{4431} = x_1 x_3^2 x_4^4 x_5 x_6^2 x_7 x_9 + \cdots .$

Skew Schur functions

Cauchy, 1815

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$
- μ fits inside λ.
- Young diagram.
  Example:
  $\lambda/\mu = (4, 4, 3, 1)/(3, 1)$
- Semistandard Young tableau (SSYT)



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$s_{4431/31} = \quad x_4^3 x_5 x_6^2 x_7 x_9 + \cdots .$

The beginning of the story

s_A: the skew Schur function for the skew shape A.

Key Facts.

- s_A is symmetric in the variables $x_1, x_2, \ldots$.
- The (non-skew) s_λ form a basis for the symmetric functions.

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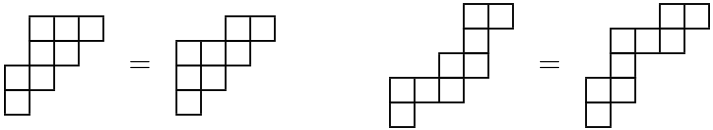
Determine necessary and sufficient conditions on shapes of A and B.



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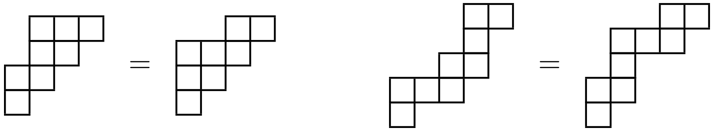

- Lou Billera, Hugh Thomas, Steph van Willigenburg (2004)
- John Stembridge (2004)
- Vic Reiner, Kristin Shaw, Steph van Willigenburg (2006)
- McN., Steph van Willigenburg (2006)
- Christian Gutschwager (2008)

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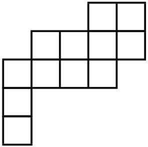
$A =$ 

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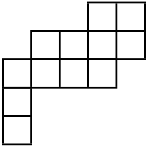
$A =$ 

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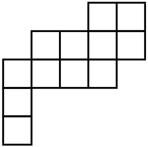
$A =$ 

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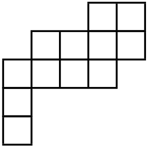
$A =$ 

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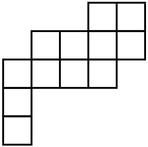
$A =$ 

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- $\text{rows}_3(A) = 11$.
- $\text{rows}_k(A) = \emptyset$ for $k > 3$.

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Theorem [RSvW, 2006]. Let A and B be skew shapes.
If $s_A = s_B$, then

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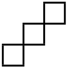
Example. $A =$ 

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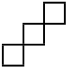
Example. $A =$ 

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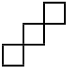
Example. $A =$ 

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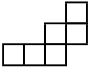
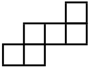
Converse is definitely not true:  $\neq$ 

Schur-positivity order

Our interest: inequalities.

Skew Schur functions are Schur-positive:

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu\nu} s_{\nu}.$$

Original Question. When is $s_{\lambda/\mu} - s_{\sigma/\tau}$ Schur-positive?

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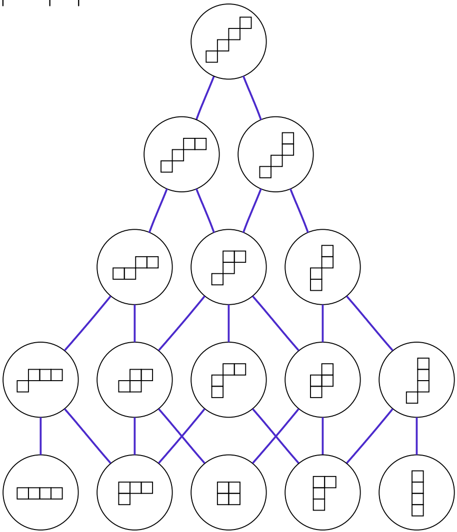
Definition. Let A, B be skew shapes. We say that

$$A \geq_s B \quad \text{if} \quad s_A - s_B \quad \text{is Schur-positive.}$$

Original goal: Characterize the Schur-positivity order $\geq_s$ in terms of skew shapes.

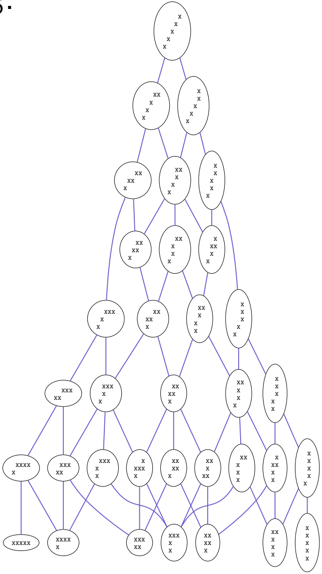
Example of a Schur-positivity poset

If $B \leq_s A$ then $|A| = |B|$.
Call the resulting ordered set P_n.
Then P_4:

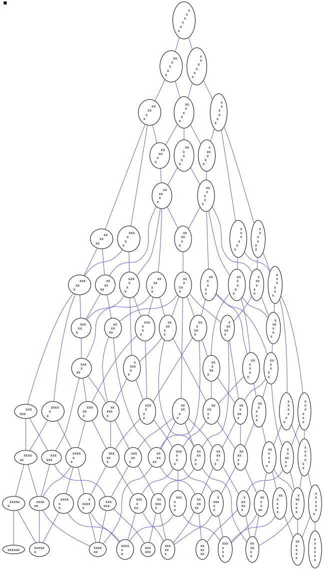


More examples

P_5:



P_6:



Known properties: Sufficient conditions

Sufficient conditions for $A \geq_s B$:

- Alain Lascoux, Bernard Leclerc, Jean-Yves Thibon (1997)
- Andrei Okounkov (1997)
- Sergey Fomin, William Fulton, Chi-Kwong Li, Yiu-Tung Poon (2003)
- Anatol N. Kirillov (2004)
- Thomas Lam, Alex Postnikov, Pavlo Pylyavskyy (2005)
- François Bergeron, Riccardo Biagioli, Mercedes Rosas (2006)
- McN., Steph van Willigenburg (2009, 2012)
- ...

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Notation. Write $\lambda \preccurlyeq \mu$ if λ is less than or equal to μ in dominance order, i.e.

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Theorem [McN. (2008)]. Let A and B be skew shapes. If $s_A - s_B$ is Schur-positive, then

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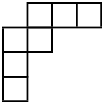
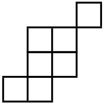
$A =$  $B =$ 

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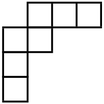
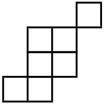
$A =$  $B =$ 

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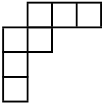
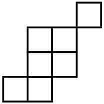
$A =$  $B =$ 

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So A and B are incomparable in Schur-positivity poset (and in "Schur support containment poset").

Summary so far

$$\boxed{s_A - s_B \text{ is Schur-pos.}} \quad \Rightarrow \quad \boxed{\text{supp}_s(A) \supseteq \text{supp}_s(B)} \quad \Rightarrow \quad \boxed{\begin{array}{l} \text{rows}_k(A) \preccurlyeq \text{rows}_k(B)\ \forall k \\ \text{cols}_\ell(A) \preccurlyeq \text{cols}_\ell(B)\ \forall \ell \\ \text{rects}_{k,\ell}(A) \leq \text{rects}_{k,\ell}(B)\ \forall k, \ell \end{array}}$$

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$\uparrow$

MIND THE GAP

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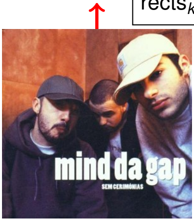

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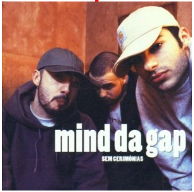

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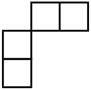
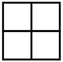
Example.

$A =$ 

$B =$ 

$s_A = s_{31} + s_{211}$

$s_B = s_{22}$

Summary so far

$s_A - s_B$ is Schur-pos. $\Rightarrow$ $\mathrm{supp}_s(A) \supseteq \mathrm{supp}_s(B)$ $\Rightarrow$

$\mathrm{rows}_k(A) \preccurlyeq \mathrm{rows}_k(B)\ \forall k$
$\mathrm{cols}_\ell(A) \preccurlyeq \mathrm{cols}_\ell(B)\ \forall \ell$
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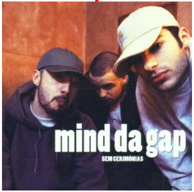
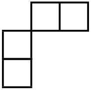
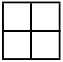
$\uparrow$



Converse is very false.

Example.

$A =$  $B =$ 

$s_A = s_{31} + s_{211}$ $\qquad$ $s_B = s_{22}$

New Goal: Find weaker algebraic conditions on A and B that imply the overlap conditions.

What algebraic conditions are being encapsulated by the overlap conditions?

Answer: use the F-basis of quasisymmetric functions

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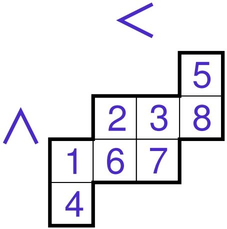
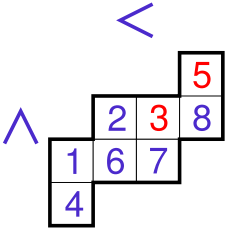
- Skew shape A.
- Standard Young tableau (SYT) T of A.



Answer: use the F-basis of quasisymmetric functions

- Skew shape A.
- Standard Young tableau (SYT) T of A.
- Descent set: $S(T) = \{3, 5\}$.



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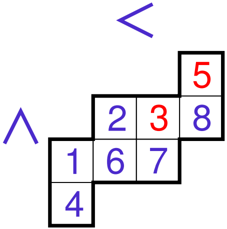

Then s_A expands in the basis of fundamental quasisymmetric functions as

$$s_A = \sum_{\text{SYT } T} F_{S(T)}.$$

Example.

$$s_{4431/31} = F_{\{3,5\}} + \cdots .$$

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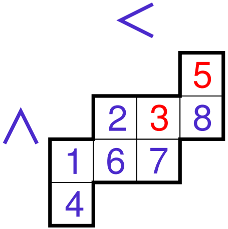

Then s_A expands in the basis of fundamental quasisymmetric functions as

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Example.

$$s_{4431/31} = F_{\{3,5\}} + \cdots .$$

Facts.

- The F form a basis for the quasisymmetric functions.
- So notions of F-positivity and F-support make sense.
- Schur-positivity implies F-positivity.
- $\text{supp}_s(A) \supseteq \text{supp}_s(B)$ implies $\text{supp}_F(A) \supseteq \text{supp}_F(B)$

New results: filling the gap

Theorem. [McN. (2013)]

$s_A - s_B$ is Schur-pos.	$\Rightarrow$	$\text{supp}_s(A) \supseteq \text{supp}_s(B)$		
$\Downarrow$		$\Downarrow$		
$s_A - s_B$ is F-positive	$\Rightarrow$	$\text{supp}_F(A) \supseteq \text{supp}_F(B)$	$\Rightarrow$	$\text{rows}_k(A) \preccurlyeq \text{rows}_k(B)\ \forall k$ $\text{cols}_\ell(A) \preccurlyeq \text{cols}_\ell(B)\ \forall \ell$ $\text{rects}_{k,\ell}(A) \leq \text{rects}_{k,\ell}(B)\ \forall k, \ell$

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Conjecture. The rightmost implication is if and only if.

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Theorem. [McN. (2013)]

$$\boxed{s_A - s_B \text{ is Schur-pos.}} \Rightarrow \boxed{\text{supp}_s(A) \supseteq \text{supp}_s(B)}$$

$$\Downarrow \qquad\qquad \Downarrow$$

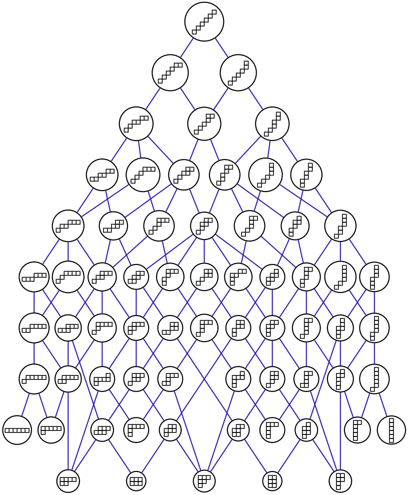
$$\boxed{s_A - s_B \text{ is } F\text{-positive}} \Rightarrow \boxed{\text{supp}_F(A) \supseteq \text{supp}_F(B)} \Longleftrightarrow \boxed{\begin{array}{l} \text{rows}_k(A) \preccurlyeq \text{rows}_k(B)\ \forall k \\ \text{cols}_\ell(A) \preccurlyeq \text{cols}_\ell(B)\ \forall \ell \\ \text{rects}_{k,\ell}(A) \leq \text{rects}_{k,\ell}(B)\ \forall k, \ell \end{array}}$$

Conjecture. The rightmost implication is if and only if.

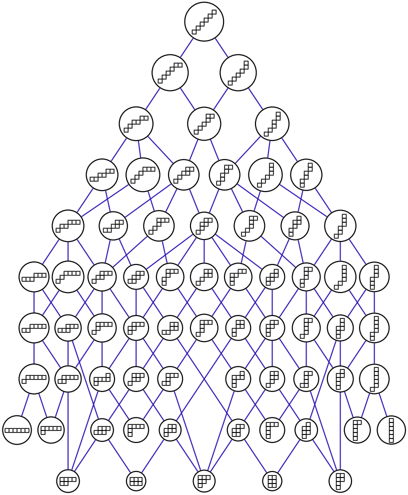
Evidence. Conjecture is true for:

- $n \leq 13$;
- horizontal strips;
- F-multiplicity-free skew shapes (as determined by Christine Bessenrodt and Steph van Willigenburg (2013));
- ribbons whose rows all have length at least 2.

$n = 6$ example



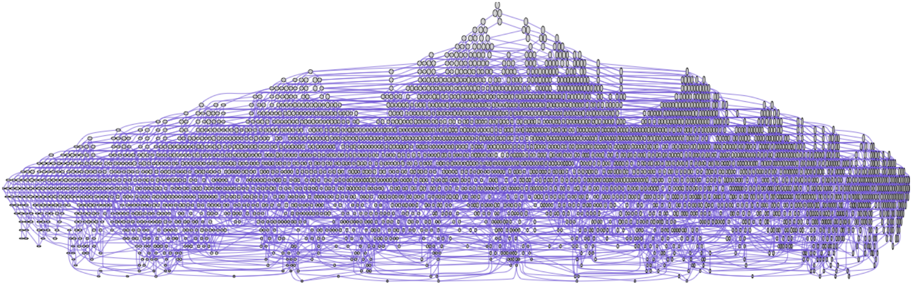
F-support containment



Dual of row overlap dominance

$n = 12$

$n = 12$ case has 12,042 edges.



$n = 13$ case has 23,816 edges.

Conclusion

$s_A - s_B$ is Schur-pos.	$\Rightarrow$	$\text{supp}_s(A) \supseteq \text{supp}_s(B)$		
$\Downarrow$		$\Downarrow$		
$s_A - s_B$ is F-positive	$\Rightarrow$	$\text{supp}_F(A) \supseteq \text{supp}_F(B)$	$\overset{?}{\Leftarrow}$ $\Rightarrow$	$\text{rows}_k(A) \preccurlyeq \text{rows}_k(B)\ \forall k$ $\text{cols}_\ell(A) \preccurlyeq \text{cols}_\ell(B)\ \forall \ell$ $\text{rects}_{k,\ell}(A) \leq \text{rects}_{k,\ell}(B)\ \forall k, \ell$

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Thanks! **Obrigado!**

Extras

$s_A - s_B$ is D-positive

$\Downarrow$

$s_A - s_B$ is Schur-pos.
$s_A - s_B$ is S-positive $\Rightarrow$ $\mathrm{supp}_D(A) \supseteq \mathrm{supp}_D(B)$, $\mathrm{supp}_s(A) \supseteq \mathrm{supp}_s(B)$, $\mathrm{supp}_S(A) \supseteq \mathrm{supp}_S(B)$

$\Downarrow$ $\Downarrow$

$s_A - s_B$ is F-positive $\Rightarrow$ $\mathrm{supp}_F(A) \supseteq \mathrm{supp}_F(B)$ $\overset{?}{\Leftarrow}$ $\Rightarrow$ $\mathrm{rows}_k(A) \preccurlyeq \mathrm{rows}_k(B)\ \forall k$, $\mathrm{cols}_\ell(A) \preccurlyeq \mathrm{cols}_\ell(B)\ \forall \ell$, $\mathrm{rects}_{k,\ell}(A) \leq \mathrm{rects}_{k,\ell}(B)\ \forall k, \ell$

$\Downarrow$ $\Downarrow$

$s_A - s_B$ is M-positive $\Rightarrow$ $\mathrm{supp}_M(A) \supseteq \mathrm{supp}_M(B)$

Extras

$s_A - s_B$ is D-positive

$\Downarrow$

$s_A - s_B$ is Schur-pos.
$s_A - s_B$ is S-positive

$\Rightarrow$

$\mathrm{supp}_D(A) \supseteq \mathrm{supp}_D(B)$
$\mathrm{supp}_s(A) \supseteq \mathrm{supp}_s(B)$
$\mathrm{supp}_S(A) \supseteq \mathrm{supp}_S(B)$

$\Downarrow$ $\Downarrow$

$s_A - s_B$ is F-positive $\Rightarrow$ $\mathrm{supp}_F(A) \supseteq \mathrm{supp}_F(B)$ $\overset{?}{\Leftarrow}$ $\Rightarrow$

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$\Downarrow$ $\Downarrow$

$s_A - s_B$ is M-positive $\Rightarrow$ $\mathrm{supp}_M(A) \supseteq \mathrm{supp}_M(B)$

Conjecture [McN., Alejandro Morales].
A quasisym skew Saturation Theorem:

$$\mathrm{supp}_F(A) \supseteq \mathrm{supp}_F(B) \quad \Longleftrightarrow \quad \mathrm{supp}_F(nA) \supseteq \mathrm{supp}_F(nB).$$

Extras

$s_A - s_B$ is D-positive				
$\Downarrow$				
$s_A - s_B$ is Schur-pos. $s_A - s_B$ is S-positive	$\Rightarrow$	$\mathrm{supp}_D(A) \supseteq \mathrm{supp}_D(B)$ $\mathrm{supp}_s(A) \supseteq \mathrm{supp}_s(B)$ $\mathrm{supp}_S(A) \supseteq \mathrm{supp}_S(B)$		
$\Downarrow$		$\Downarrow$		
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$\Downarrow$		$\Downarrow$		
$s_A - s_B$ is M-positive	$\Rightarrow$	$\mathrm{supp}_M(A) \supseteq \mathrm{supp}_M(B)$		

Conjecture [McN., Alejandro Morales].

A quasisym skew Saturation Theorem:

$$\mathrm{supp}_F(A) \supseteq \mathrm{supp}_F(B) \iff \mathrm{supp}_F(nA) \supseteq \mathrm{supp}_F(nB).$$

Thanks! Obrigado!