Revisiting Hartle's model using perturbed matching theory to second order: amending the change in mass

Based on
Borja Reina and Raül Vera (UPV/EHU) *accepted at Class.Quant.Grav.*
and Marc Mars (U. Salamanca) *Class.Quant.Grav. (2005)*

University of the Basque Country UPV/EHU

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Literature

- Fluid balls in "slow rotation" approximation in equilibrium (stationary perturbations)

- *Hartle (1967)*: first and second order stationary and axisymmetric perturbations of static perfect-fluid balls in vacuum.
- More recent (analytic) works on models for compact objects in equilibrium: *Bradley et al. (2007)*, and more, and *Cabezas et al. (2007)*, *Blázquez-Salcedo et al. (2012)*, *Cuchi et al. (2013)*, ...

Consistent/rigorous (*) **matching perturbation theory** :
first order *Battye, Carter (1995)* and *Mukohyama (2000)* (almost) and
second order *Mars (2005)* in full generality.

More literature on linearised perturbed matching: *Cunningham, Price, Moncrief (1978,79), Gerlach, Sengupta (1979); Martín-García, Gundlach (2001); and Brizuela et al. (2010) for higher orders*

(*) *Mars, Mena, Vera (2007)*

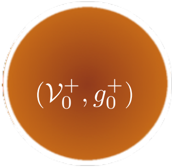
The setting: Hartle's model for rotating stars in GR

Static and spherically symmetric star

Global model of a (spher. symm.) non-rotating star

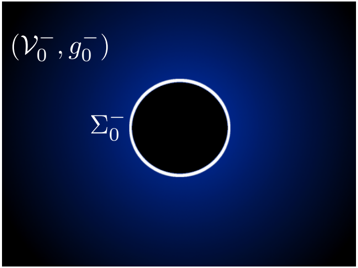
Fluid ball (interior)

- Eqs. for a perfect fluid: $E(r_+)$, $P(r_+)$ + Barotropic EOS
- $\Rightarrow$ given $E(0) = E_c$, E and P are integrated.



Σ_0^+

Asymptotically flat vacuum (exterior): Schwarzschild



$$g_0^\pm = -e^{\nu^\pm(r_\pm)} dt_\pm^2 + e^{\lambda^\pm(r_\pm)} dr_\pm^2 + r_\pm^2 (d\theta_\pm^2 + \sin^2\theta_\pm d\varphi_\pm^2)$$

$$\Sigma_0^\pm = \{r_\pm = a_\pm\}, \quad \vec{n}^\pm = -e^{-\frac{\lambda^\pm(a_\pm)}{2}} \partial_{r_\pm}|_{\Sigma_0^\pm}$$

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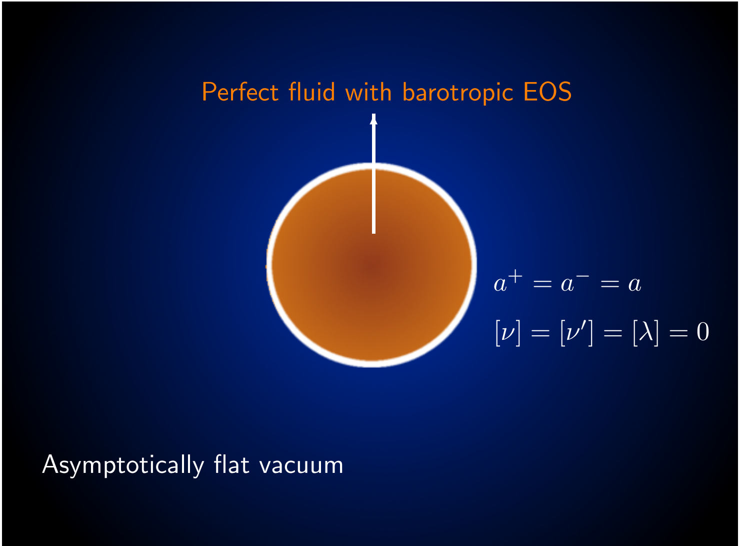

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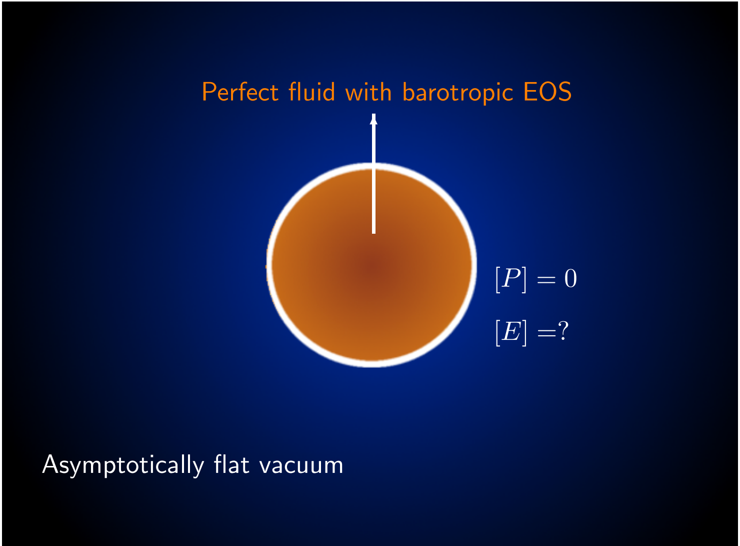

The setting: Hartle's model for rotating stars in GR

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Match the $+$ and $-$ regions to obtain a global model of the star:

Because of $[\nu] = [\lambda] = 0$ (apart from $[\nu'] = 0$) the use of

$$g_0 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

for some $r \in (0, \infty)$ is standard in many works. The functions in the metric are said to be "continuous".

However, extending such "continuity" to other settings, in general, can lead to wrong conclusions. For instance, extending to a perturbative scheme.

In particular, it does in Hartle's perturbative setting.

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The setting: Hartle's model for rotating stars in GR

"Slow" rotation

Starting from the spherically symm. and static configuration (**background**), stationary and axially symmetric perturbations are introduced to describe "slow" rotation in equilibrium.

Quantities that arise as a consequence of rotation:

- J: Angular momentum
- A (in Hartle's notation): Proportional to the quadrupolar moment and related to the ellipticity of the star
- δM: Change in mass of the rotating configuration, with respect to the static one, needed to keep the central density of the star E_c unchanged.

In Hartle's model, these constants are calculated joining the fluid and the vacuum regions **assuming the "continuity of the metric"** in some system of coordinates used.

The setting: Hartle's model for rotating stars in GR

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δM: Change in mass

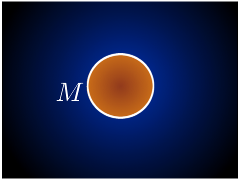
- Change in mass of the rotating configuration needed to keep the central density of the star E_c unchanged.
- The assumed "continuity of the metric" in Hartle's model (in those coordinates) is not valid to calculate this constant.
- $\delta M = a e^{-\lambda(a)} m_0^+(a) + \frac{J^2}{a^3} + 4\pi \frac{a^3}{M} E(a) \tilde{\mathcal{P}}_0(a)$

The setting: Hartle's model for rotating stars in GR

Excess of mass δM

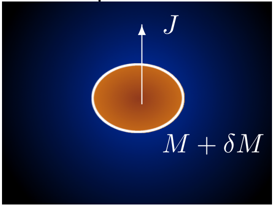
Static star

- Integrate the equations (TOV) with a fixed central energy density E_c
- The mass M is determined



Rotating star

- Integrate the field equations for the perturbations with the same E_c that in the static case.
- The star has a mass M plus a contribution of the second order rotational perturbations



The setting: Hartle's model for rotating stars in GR

Perturbative setting: Remarks

Explicit assumptions

- Barotropic equation of state.
- Stationary model.
- Axial and equatorial symmetry.
- Rigid rotation.

Implicit assumptions

- Absence of convective motions.
- Explicit global coordinates in which the metric is at least C^0.

Hartle's model

Stationary and axially symmetric spacetime: $\vec{\xi}$ and $\vec{\eta}$.
Matter content of the interior: perfect fluid: $\hat{E}$, $\hat{P}$, fluid flow $\vec{u}$.
Exterior: vacuum

- Perturbation parameter Ω defined as $\vec{u} \propto \vec{\xi} + \Omega\vec{\eta}$ (rigid. rot.)

In Hartle's model **the second order metric** is:

$$ds^2 = -\, e^{\nu(r)}(1+2h(r,\theta))dt^2 + e^{\lambda(r)}\left(1+\frac{2m(r,\theta)}{r-2M}\right)dr^2$$
$$+\, r^2\,(1+2k(r,\theta))\left(d\theta^2+\sin^2\theta(d\varphi-\omega(r,\theta)dt)^2\right)\;,\; \boxed{r\in(0,\infty)}$$

- Background functions: $\nu(r)$, $\lambda(r)$.
- 1st order: $\omega(r,\theta)$. Regular origin + asymp. flatness $\Rightarrow \omega(r)$.
- 2nd order: $h(r,\theta)$,$m(r,\theta)$,$k(r,\theta)$ (at least C^0)
  - Surface of the star determined by: $r = a+\xi(a,\theta)$, where $\hat{P}(r+\xi(r,\theta),\theta) = P(r)$, so that $\hat{P}(a+\xi(a,\theta),\theta) = 0$.

Hartle's model: 2nd order (I)

Metric at second order: $h(r,\theta)$, $m(r,\theta)$ and $k(r,\theta)$.

Using the decompositions $h(r,\theta) = \sum_{l=0}^{\infty} h_l(r) P_l(\cos\theta)$, etc...

in Hartle's work it is argued that since

- For $l > 2$: homogeneous equations (no sources from ω)
- Equatorial symmetry (only even l's)

then

$$
\begin{aligned}
h(r,\theta) &= h_0(r) + h_2(r)P_2(\cos\theta) \\
m(r,\theta) &= m_0(r) + m_2(r)P_2(\cos\theta) \\
k(r,\theta) &= k_0(r) + k_2(r)P_2(\cos\theta) \\
\Rightarrow \qquad \xi(r,\theta) &= \xi_0(r) + \xi_2(r)P_2(\cos\theta)
\end{aligned}
$$

Hartle's model: 2nd order (II)

Field equations for the interior and BCs provide:

$l = 0$ **problem:** change in mass:

Interior $l = 0$ problem:

- Hydrostatic equilibrium first integral $\gamma - h_0 = (...)\xi_0 + (\text{rotation})^2$
- 1st order inhomogeneous system of ODE's for m_0 and ξ_0
- BC: imposed at the origin on ξ_0 and m_0 to keep E_c unchanged
- $\Rightarrow$ obtain the values $\xi_0(a)$ and $m_0(a)$

Exterior AF vacuum $l = 0$ problem:

$$\rightarrow \qquad m_0^-(r) = \delta M - \frac{J^2}{r^3}, \qquad h_0^-(r) = -\frac{1}{r-2M}\left(\delta M - \frac{J^2}{r^3}\right)$$

for some constant δM.

Matching:

Continuity of m_0 at $r = a \Rightarrow \qquad \delta M = m_0(a) + \frac{J^2}{a^3}$

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Hartle's model: 2nd order (II)

Field equations for the interior and BCs provide

$l = 2$ problem: Shape

- Hydrostatic equilibrium first integral
  $0 = h_2 + (...)\xi_2 + (\text{rotation})^2$
- Algebraic equation for m_2
- 1st order inhomogeneous system of ODE's for h_2 and k_2
- BC: regularity at the origin for both h_2, k_2
  and $h_2, k_2 \to 0$ at infinity (AF)
- Matching: Continuity for h_2 and k_2 at $r = a$.
- $\xi_2 = \xi_2(a, M, h_2, \Omega, \omega) \Rightarrow \epsilon = -3\xi_2(a)/2a$

Revisiting Hartle's model

Explicit assumptions

Barotropic equation of state
Stationary model
Axial and equatorial symmetry
Rigid rotation

Implicit assumptions

- Absence of convective motions.
- Explicit global coordinates in which the metric is, at least, C^0.

Our work: global aim and (one) result

- Put the model on firm grounds (given a consistent theory of perturbed matchings to second order)
- Result: that assumption is not consistent: $[m_0] \neq 0$ in general. In fact, the resulting expression for the change in mass δM computed is not correct.

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Perturbed matching conditions to second order

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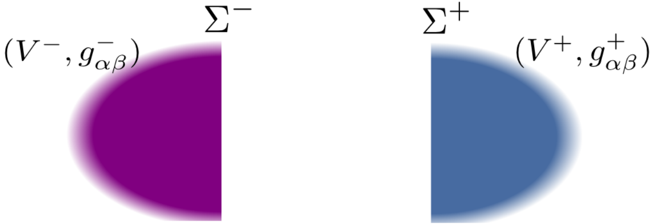

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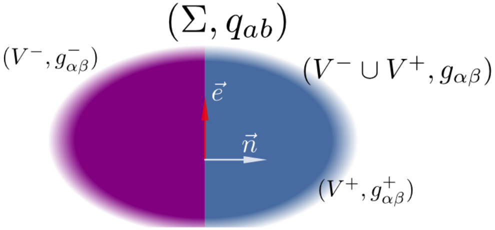

Properly matched spacetimes $(V^-, g^-_{\alpha\beta})$ and $(V^+, g^+_{\alpha\beta})$ across $\Sigma^- = \Sigma^+ (\equiv \Sigma)$.

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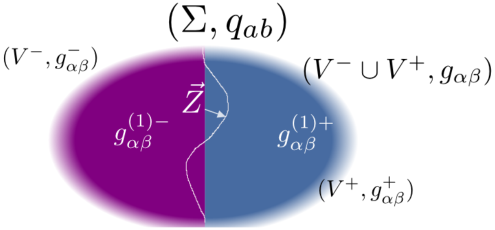

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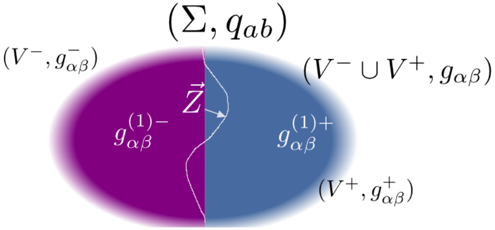

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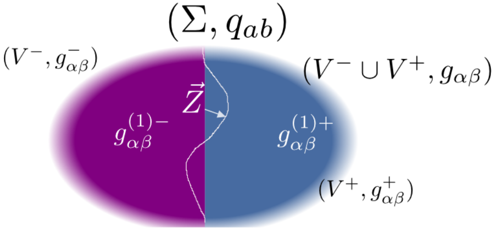

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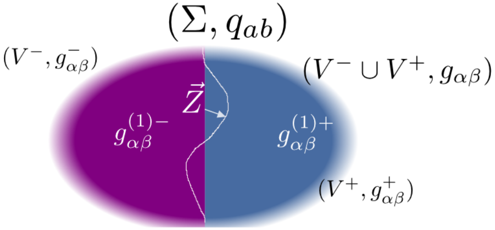

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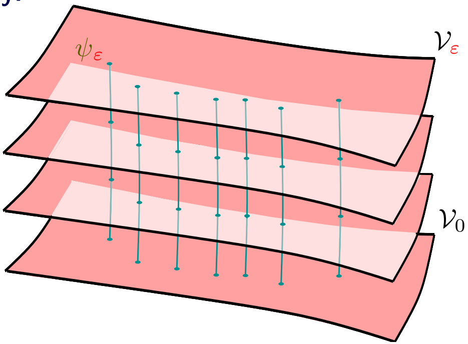
We are going to go to second order: $g^{(2)-}_{\alpha\beta}$, $g^{(2)+}_{\alpha\beta}$, $\vec{Z}^{\pm}_2$.

First and second order perturbations

Perturbation theory:



- One parameter family of spacetimes $(\mathcal{V}_\varepsilon, \hat{g}_\varepsilon)$, with diffeomorphically identified points, through $\psi_\varepsilon : \mathcal{V}_0 \to \mathcal{V}_\varepsilon$.
- Background chosen at $\varepsilon = 0 : (\mathcal{V}_0, g)$, with $g \equiv \hat{g}_0$
- Define the family of tensors g_ε on $\mathcal{V}_0$ by $g_\varepsilon \equiv \psi^*_\varepsilon(\hat{g}_\varepsilon)$

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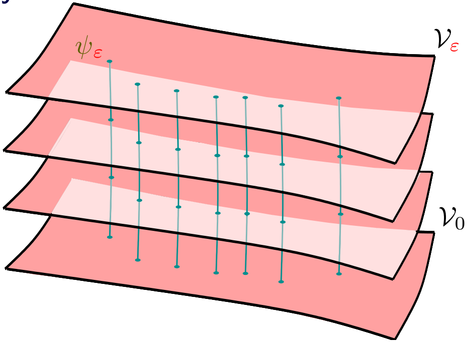

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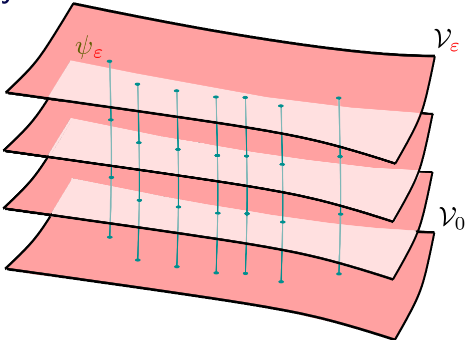

Family of tensors g_ε on $(\mathcal{V}_0, g)$ such that $g = g_0$
Metric perturbations: symmetric tensors defined on $(\mathcal{V}_0, g)$

$$K_1 = \left.\frac{\partial g_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} (= g^{(1)}), \qquad K_2 = \left.\frac{\partial^2 g_\varepsilon}{\partial \varepsilon^2}\right|_{\varepsilon=0} (= g^{(2)})$$

First and second order perturbations

Perturbation theory: is the study of tensor fields K_1 and K_2 satisfying certain **field equations on a fixed background** $(\mathcal{V}_0, g)$.

The **field equations for** K_1 and K_2 come from imposing that g_{ε} satisfy the same field equations as the background.

Linearised vacuum field equations:

Background: $R_{\alpha\beta}(g) = 0$.

We need to impose $\left.\frac{\partial R_{\alpha\beta}(g_{\varepsilon})}{\partial \varepsilon}\right|_{\varepsilon=0} = 0$ and $\left.\frac{\partial^2 R_{\alpha\beta}(g_{\varepsilon})}{\partial \varepsilon^2}\right|_{\varepsilon=0} = 0,$

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Relation between Riemann tensors of two arbitrary metrics g_0 and g_ε:

$R^{\alpha}_{\beta\gamma\delta}(g_\varepsilon) = R^{\alpha}_{\beta\gamma\delta}(g_0) + 2\nabla^0_{[\gamma} C^{\alpha}_{\delta]\beta} + 2C^{\alpha}_{\rho[\gamma} C^{\rho}_{\delta]\beta},$

where $C^{\alpha}_{\gamma\beta} = \frac{1}{2} g_\varepsilon^{\alpha\mu} (\nabla^0_\beta g_{\varepsilon\mu\gamma} + \nabla^0_\gamma g_{\varepsilon\mu\beta} - \nabla^0_\mu g_{\varepsilon\gamma\beta})$

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$$\left.\frac{\partial R_{\alpha\beta}(g_\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{1}{2}(2\nabla_\mu \nabla_{(\alpha} K^\mu_{1\beta)} - \nabla_\mu \nabla^\mu K_{1\alpha\beta} - \nabla_\alpha \nabla_\beta K^\mu_{1\mu})$$

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1st order vacuum field equation:

$$L(K_1) \equiv \tfrac{1}{2}(2\nabla_\mu \nabla_{(\alpha} K^{\mu}_{1\beta)} - \nabla_\mu \nabla^\mu K_{1\alpha\beta} - \nabla_\alpha \nabla_\beta K^{\mu}_{1\mu}) = 0$$

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2nd order vacuum field equation:

$L(K_2) +$ quadratic terms in $(K_1, \nabla K_1) = 0$

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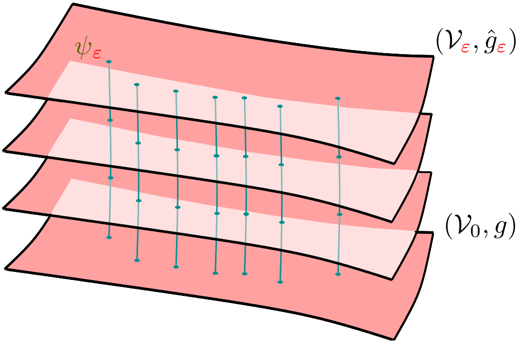
$L(K_2) +$ quadratic terms in $(K_1, \nabla K_1) = 0$

Non-vacuum $T_{\alpha\beta} \neq 0$, then 0 is substituted by the appropriate perturbations of the matter fields, using $T_{\alpha\beta}(\varepsilon)$.

First and second order perturbations

Perturbation theory: inherent gauge freedom



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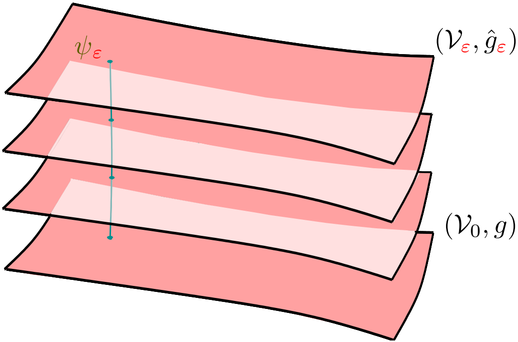
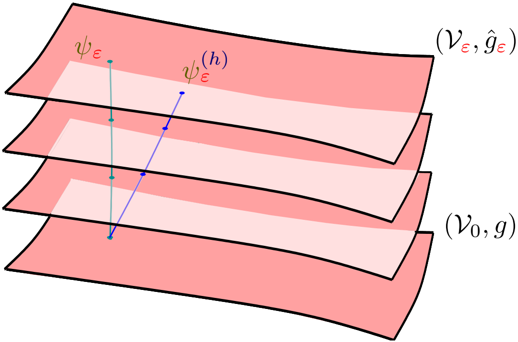


First and second order perturbations

Perturbation theory: inherent gauge freedom

Taking a different ψ means a **different gauge:** say $\psi^{(h)}$



First and second order perturbations

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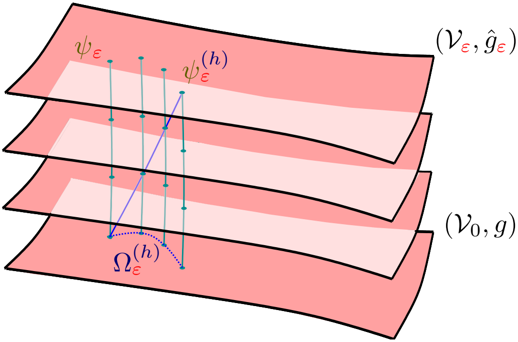

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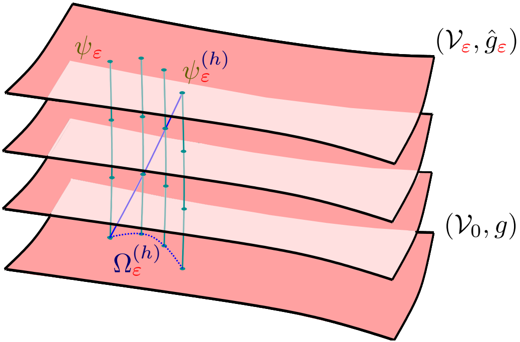

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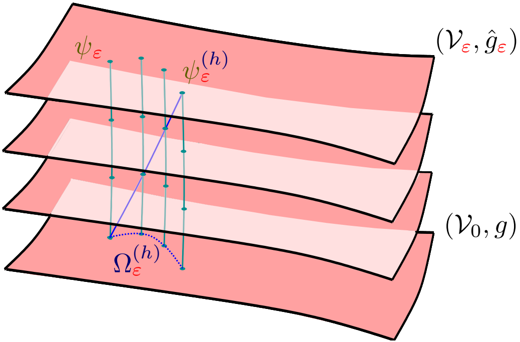

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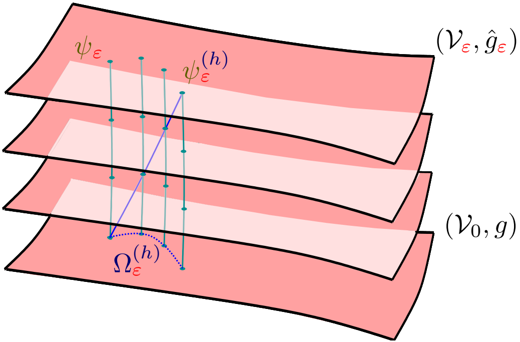


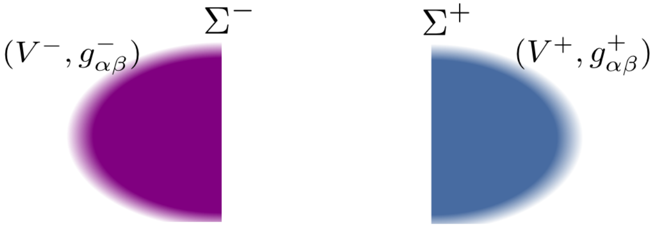
Define: $\vec{s}_1 \equiv \left.\frac{\partial \Omega_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0}$, $\vec{s}_2 \equiv \left.\left.\frac{\partial}{\partial \epsilon}\frac{\partial(\Omega_{\varepsilon+\epsilon}\circ\Omega_\epsilon^{-1})}{\partial \varepsilon}\right|_{\varepsilon=0}\right|_{\epsilon=0} + \nabla_{\vec{s}_1}\vec{s}_1$, then

$$K_1^{(h)} = K_1 + \mathcal{L}_{\vec{s}_1} g$$ **(Bruni et al. (1997))**

$$K_2^{(h)}{}_{\alpha\beta} = K_{2\alpha\beta} + \mathcal{L}_{\vec{s}_2} g_{\alpha\beta} + 2\mathcal{L}_{\vec{s}_1} K_{2\alpha\beta} + -2 s_1^\mu s_1^\nu R_{\alpha\mu\beta\nu} + 2\nabla_\alpha s_1^\mu \nabla_\beta s_{1\mu}$$

Standard (Darmois) matching conditions

We are given **two spacetimes with (no null) boundary**:



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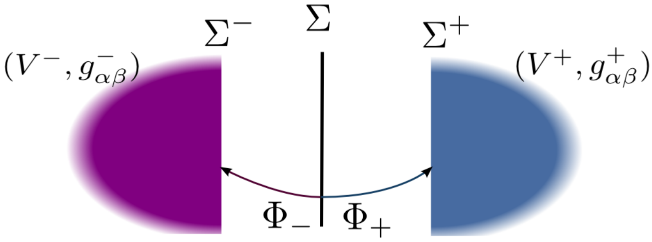

1. **Gluing:** identify boundaries through $\Phi_+ \circ \Phi_-^{-1}$: $\Sigma^+ = \Sigma^- (\equiv \Sigma)$

   embeddings: $\Phi_\pm : \Sigma \to \Sigma^\pm (\xi^a \to x^\alpha_\pm = \Phi^\alpha_\pm(\xi^a))$

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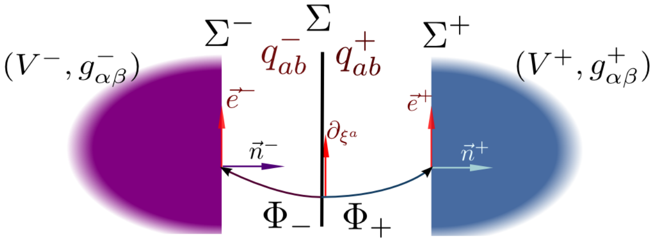

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2. **First fundamental forms:** $q^\pm_{ab} \equiv \Phi^*_\pm g^\pm_{\alpha\beta}$

   Push-forwards: $\mathrm{d}\Phi(\partial_{\xi^a}) = \frac{\partial \Phi^\alpha}{\partial \xi^a} \partial_{x^\alpha} \equiv \vec{e}_a = e_a{}^\alpha \partial_{x^\alpha}$,
   Unit normals: $\boldsymbol{n}(\vec{e}_a) = 0$
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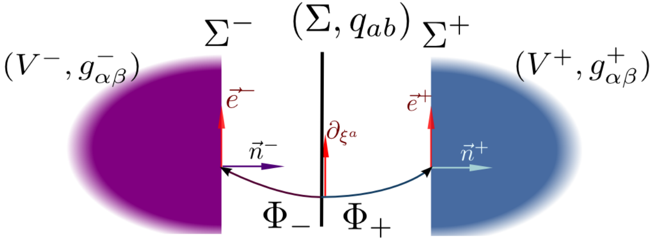

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There $\exists$ a C^0 metric g on $V^- \cup V^+$ with $g|_{V^\pm} = g^\pm$ iff $q^+_{ab} = q^-_{ab}$

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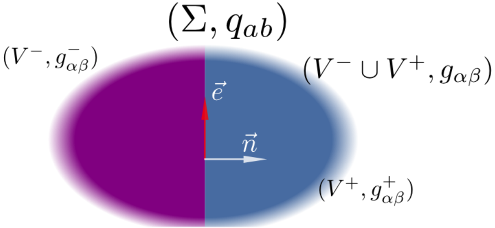

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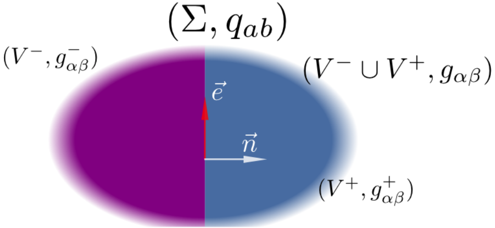

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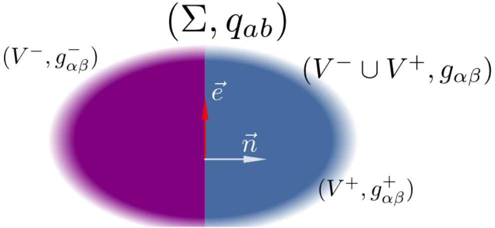

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The distributional Riemann tensor of g contains no Dirac δ part with support on Σ iff $\kappa^+_{ab} = \kappa^-_{ab}$

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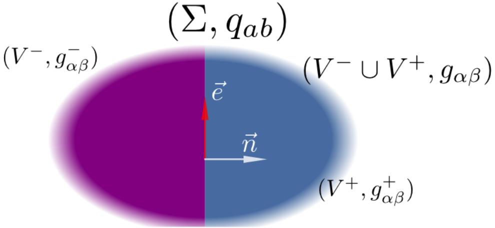

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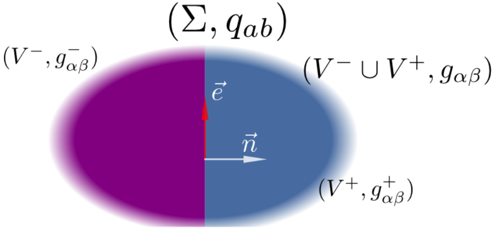

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For two families $(\mathcal{V}^-_\varepsilon, \hat{g}^-_\varepsilon, \Sigma^-_\varepsilon)$ and $(\mathcal{V}^+_\varepsilon, \hat{g}^+_\varepsilon, \Sigma^+_\varepsilon)$ we get a family of diff. related Σ_ε ($\Rightarrow$ diff. related to Σ_0), and the corresponding q^+_ε, q^-_ε, κ^+_ε and κ^-_ε, and matching equations $q^+_\varepsilon = q^-_\varepsilon$, $\kappa^+_\varepsilon = \kappa^-_\varepsilon$

Standard (Darmois) matching conditions

Take a **matched background configuration:**
$(\mathcal{V}_0^+, g_0^+)$ matched to $(\mathcal{V}_0^-, g_0^-)$ accross $\Sigma_0^+ = \Sigma_0^- \equiv \Sigma_0$:

The linearised matching conditions are just

$$\partial_\varepsilon q_\varepsilon^+|_{\varepsilon=0} = \partial_\varepsilon q_\varepsilon^-|_{\varepsilon=0}$$
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And, to second order

$$\partial_\varepsilon^2 q_\varepsilon^+|_{\varepsilon=0} = \partial_\varepsilon^2 q_\varepsilon^-|_{\varepsilon=0}$$
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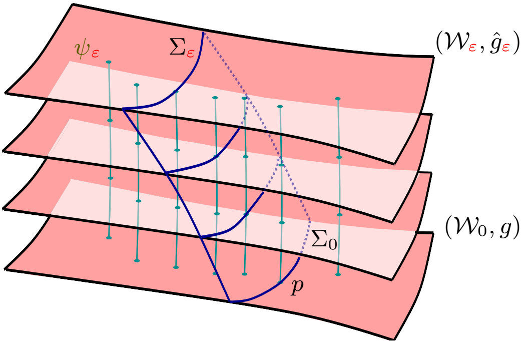
We want to write these equations in terms of $K_1^\pm|_{\Sigma^\pm}$ (and $K_2^\pm|_{\Sigma^\pm}$) and **background objects only**. Recall these will be **equations in** Σ_0.

How to construct the tensors q_ε and κ_ε:

First and second order deformations of hypersurfaces

For $\pm$: take $\mathcal{V}_\varepsilon$ as a submanifold with boundary Σ_ε in a larger $\mathcal{W}_\varepsilon$.



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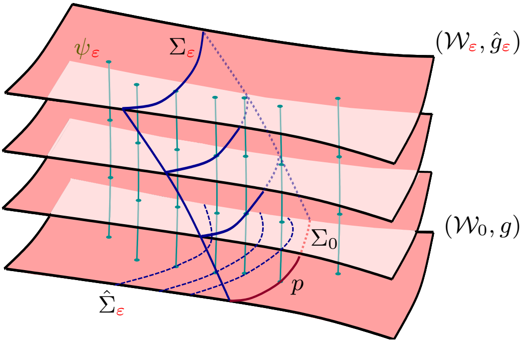
$\hat{\Sigma}_\varepsilon = \psi_\varepsilon^{-1}(\Sigma_\varepsilon)$



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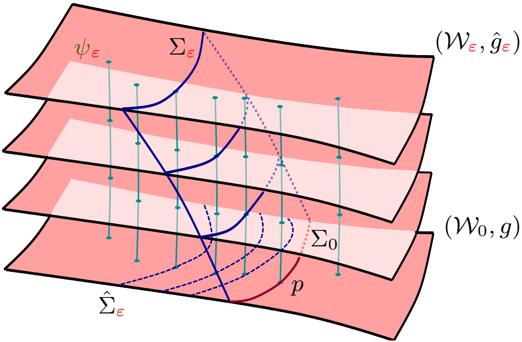

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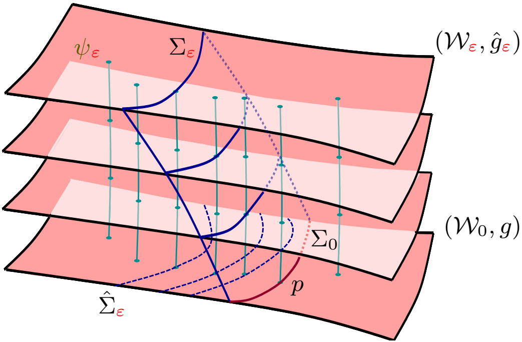

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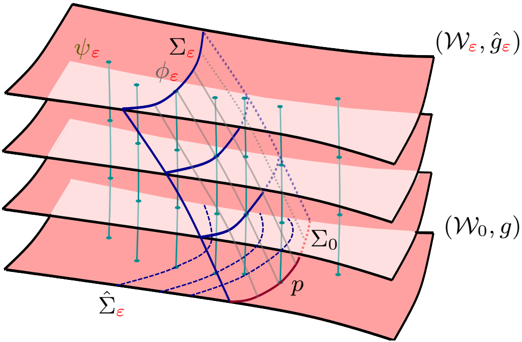

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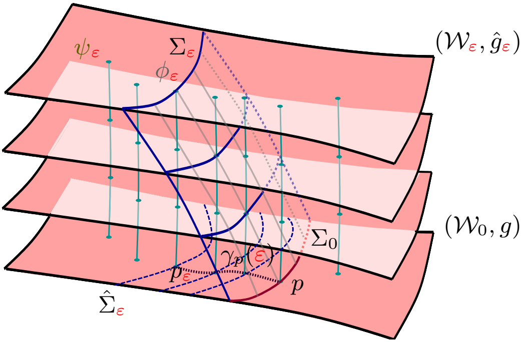

The composition of $\psi_\varepsilon^{-1} \circ \phi_\varepsilon$ provides, for any $p \in \Sigma_0$ a path $\gamma_p(\varepsilon) \subset \mathcal{W}_0$.

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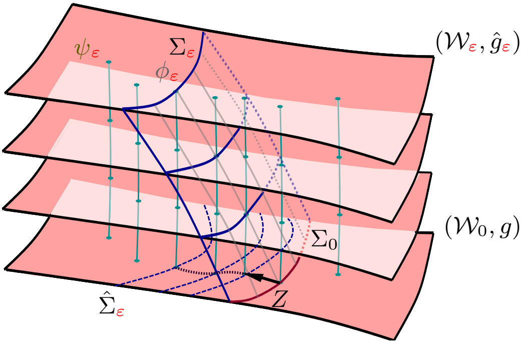

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The tangent of $\gamma_p(\varepsilon)$ defines a vector $\vec{Z}_1$ at points on Σ_0. In terms of the coordinated embedding $\Phi_\varepsilon \equiv \psi_\varepsilon^{-1} \circ \phi_\varepsilon$ it reads $Z_1^\alpha(\xi^a) = \partial_\varepsilon \Phi^\alpha(\xi^a, \varepsilon)|_{\varepsilon=0}$
And the acceleration $\vec{Z}_2$:
$Z_2^\alpha(\xi^a) = \partial_\varepsilon^2 \Phi^\alpha(\xi^a, \varepsilon)|_{\varepsilon=0} + \Gamma^\alpha_{\beta\gamma}(x_0(\xi^a)) Z_1^\beta(\xi^a) Z_1^\gamma(\xi^a)$

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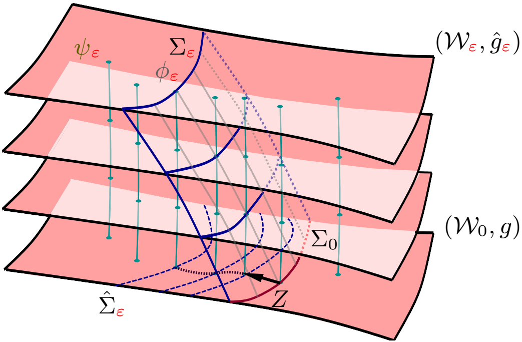

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$\Rightarrow \vec{Z}_1, \vec{Z}_2$

By construction, under **spacetime gauges** $\vec{s}_1$ and $\vec{s}_2$, $\vec{Z}$'s transform as

$$\vec{Z}_1^{(h)} = \vec{Z}_1 - \vec{s}_1$$

$$\vec{Z}_2^{(h)} = \vec{Z}_2 - \vec{s}_2 - 2\nabla_{\vec{Z}_1}\vec{s}_1 + 2\nabla_{\vec{s}_1}\vec{s}_1$$

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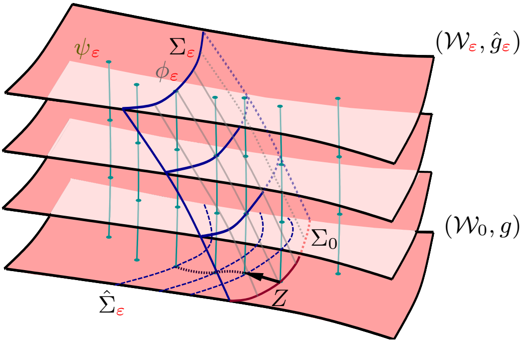

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$\Rightarrow \vec{Z}_1, \vec{Z}_2$

We can decompose $Z^\alpha = Qn^\alpha + T^\alpha|_{\Sigma_0}$,

and take instead quantities defined in Σ_0: Q and T^a $\quad Q \leftrightarrow \Phi^* Q,\ T^\alpha = \mathrm{d}\Phi(T^a)$

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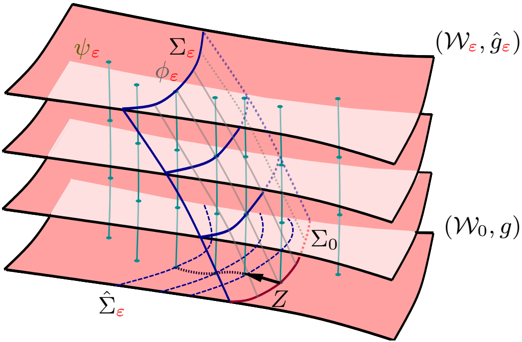

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Deformation vectors $\vec{Z}_1^\pm$ and $\vec{Z}_2^\pm$ (at either side $\pm$)

$\vec{Z}_1^\pm \to Q_1^\pm, T_1^{a\,\pm}, \qquad \vec{Z}_2^\pm \to Q_2^\pm, T_2^{a\,\pm}$

By construction, Q's and T^a's **depend** on the **spacetime gauges** $\psi^\pm$, and T^a's also depend on the **hypersurface gauge** ϕ (**but not** Q's)

First order perturbed matching conditions

Take **the background configuration**: the spacetime $\mathcal{V}_0^{\pm}$ with metrics $g^{\pm}$, the embeddings $\Phi^{\pm}$ from a timelike (codim 1) Σ $(= \Sigma_0)$, $e_a^{\alpha} = \mathrm{d}\Phi(\partial_a)$, and corresponding (unit) normals $n_{\alpha}^{\pm}|_{\Sigma^{\pm}}$.

Ingredients: to compute the 1st order perturbations of the first and second fund. forms: $q^{(1)} \equiv \partial_{\varepsilon} q_{\varepsilon}|_{\varepsilon=0}$ and $\kappa^{(1)} \equiv \partial_{\varepsilon} \kappa_{\varepsilon}|_{\varepsilon=0}$:

- **Perturbed metric tensor** K_1
- **1st order deformation vector** of Σ (unknown): $\vec{Z}_1 \to Q_1, \vec{T}_1$

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- **Perturbed metric tensor** $\qquad K_1$
- **1st order deformation vector** of Σ (unknown): $\qquad \vec{Z}_1 \to Q_1, \vec{T}_1$

Theorem (Battye, Carter (1995)): perturbations of hypersurfs.

$$
\begin{aligned}
q_{ab}^{(1)} &= \mathcal{L}_{\vec{T}_1} q_{ab} + 2Q_1 \kappa_{ab} + e_a^{\alpha} e_b^{\beta} K_{1\,\alpha\beta}|_{\Sigma},\\
\kappa_{ab}^{(1)} &= \mathcal{L}_{\vec{T}_1} \kappa_{ab} - D_a D_b Q_1 + Q_1 (n^{\mu} n^{\nu} R_{\alpha\mu\beta\nu} e_a^{\alpha} e_b^{\beta} + \kappa_{ac} \kappa^{c}{}_{b})\\
&+ \frac{1}{2} K_{1\,\alpha\beta} n^{\alpha} n^{\beta} \kappa_{ab} - n_{\mu} S^{(1)\mu}_{\alpha\beta} e_a^{\alpha} e_b^{\beta}|_{\Sigma},
\end{aligned}
$$

First order perturbed matching conditions

Take **the background configuration**: the spacetime $\mathcal{V}_0^{\pm}$ with metrics $g^{\pm}$, the embeddings $\Phi^{\pm}$ from a timelike (codim 1) Σ $(= \Sigma_0)$, $e_a^{\alpha} = \mathrm{d}\Phi(\partial_a)$, and corresponding (unit) normals $n_{\alpha}^{\pm}|_{\Sigma^{\pm}}$.

Ingredients: to compute the 1st order perturbations of the first and second fund. forms: $q^{(1)} \equiv \partial_{\varepsilon} q_{\varepsilon}|_{\varepsilon=0}$ and $\kappa^{(1)} \equiv \partial_{\varepsilon} \kappa_{\varepsilon}|_{\varepsilon=0}$:

- **Perturbed metric tensor** K_1
- **1st order deformation vector** of Σ (unknown): $\vec{Z}_1 \to Q_1, \vec{T}_1$

Theorem (Battye, Carter (1995)): perturbations of hypersurfs.

$$
\begin{aligned}
q_{ab}^{(1)} &= \mathcal{L}_{\vec{T}_1} q_{ab} + 2Q_1 \kappa_{ab} + e_a^{\alpha} e_b^{\beta} K_{1\,\alpha\beta}|_{\Sigma}, \\
\kappa_{ab}^{(1)} &= \mathcal{L}_{\vec{T}_1} \kappa_{ab} - D_a D_b Q_1 + Q_1 (n^{\mu} n^{\nu} R_{\alpha\mu\beta\nu} e_a^{\alpha} e_b^{\beta} + \kappa_{ac} \kappa^{c}{}_{b}) \\
&+ \frac{1}{2} K_{1\,\alpha\beta} n^{\alpha} n^{\beta} \kappa_{ab} - n_{\mu} S^{(1)\mu}_{\alpha\beta} e_a^{\alpha} e_b^{\beta}|_{\Sigma},
\end{aligned}
$$

where D_a is the three dimensional covariant derivative of (Σ, q_{ab}) and

$$
2S^{(1)\alpha}_{\beta\gamma} \equiv \nabla_{\beta} K_1{}^{\alpha}_{\gamma} + \nabla_{\gamma} K_1{}^{\alpha}_{\beta} - \nabla^{\alpha} K_{1\,\beta\gamma}
$$

First order perturbed matching conditions

Take **the background configuration**: the spacetime $\mathcal{V}_0^{\pm}$ with metrics $g^{\pm}$, the embeddings $\Phi^{\pm}$ from a timelike (codim 1) Σ $(= \Sigma_0)$, $e_a^{\alpha} = \mathrm{d}\Phi(\partial_a)$, and corresponding (unit) normals $n_{\alpha}^{\pm}|_{\Sigma^{\pm}}$.

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Theorem (Battye, Carter (1995)): perturbations of hypersurfs.

$$
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q_{ab}^{(1)} &= \mathcal{L}_{\vec{T}_1} q_{ab} + 2Q_1 \kappa_{ab} + e_a^{\alpha} e_b^{\beta} K_{1\,\alpha\beta}|_{\Sigma}, \\
\kappa_{ab}^{(1)} &= \mathcal{L}_{\vec{T}_1} \kappa_{ab} - D_a D_b Q_1 + Q_1 (n^{\mu} n^{\nu} R_{\alpha\mu\beta\nu} e_a^{\alpha} e_b^{\beta} + \kappa_{ac} \kappa^{c}{}_{b}) \\
&+ \frac{1}{2} K_{1\,\alpha\beta} n^{\alpha} n^{\beta} \kappa_{ab} - n_{\mu} S^{(1)\mu}_{\alpha\beta} e_a^{\alpha} e_b^{\beta}|_{\Sigma},
\end{aligned}
$$

Theorem (Mars (2005)): 1st order matching conditions are fulfilled:

iff $\exists\ Q_1^{\pm}$ and $\vec{T}_1^{\pm}$ such that $q_{ab}^{(1)+} = q_{ab}^{(1)-}, \quad \kappa_{ab}^{(1)+} = \kappa_{ab}^{(1)-}$

Second order perturbed matching conditions

Take the **first order matched configuration**.

Ingredients: to compute the 2nd order perturbations of the first and second fund. forms: $q^{(2)} \equiv \partial_\varepsilon^2 q_\varepsilon|_{\varepsilon=0}$ and $\kappa^{(2)} \equiv \partial_\varepsilon^2 \kappa_\varepsilon|_{\varepsilon=0}$:

- **Perturbed metric tensor** K_2
- **1st order deformation vector** of Σ (unknown): $\vec{Z}_2 \to Q_2, \vec{T}_2$

Second order perturbed matching conditions

Take the **first order matched configuration**.

Ingredients: to compute the 2nd order perturbations of the first and second fund. forms: $q^{(2)} \equiv \partial_\varepsilon^2 q_\varepsilon|_{\varepsilon=0}$ and $\kappa^{(2)} \equiv \partial_\varepsilon^2 \kappa_\varepsilon|_{\varepsilon=0}$:

- **Perturbed metric tensor** K_2
- **1st order deformation vector** of Σ (unknown): $\vec{Z}_2 \to Q_2, \vec{T}_2$

Theorem (Mars (2005)): perturbations of hypersurfs.

$$
\begin{aligned}
q^{(2)}_{ab} &= \mathcal{L}_{\vec{T}_2} q_{ab} + 2Q_2\kappa_{ab} + K_{2\alpha\beta} e^\alpha_a e^\beta_b + 2\mathcal{L}_{\vec{T}_1} q^{(1)}_{ab} - \mathcal{L}_{\vec{T}_1}\mathcal{L}_{\vec{T}_1} q_{ab} + \\
&+ \mathcal{L}_{2Q_1\vec{\tau}' - 2Q_1\kappa(\vec{T}_1) - D_{\vec{T}_1}\vec{T}_1} q_{ab} + 2D_aQ_1D_bQ_1 \\
&+ 2\left(T_1{}^c T_1{}^d \kappa_{cd} - 2\vec{T}_1(Q_1) + 2Q_1Y'\right)\kappa_{ab} + \\
&+ 2Q_1^2\left(-n^\mu n^\nu R_{\alpha\mu\beta\nu} e^\alpha_a e^\beta_b + \kappa_{ac}\kappa^c_b\right) - 4Q_1 n_\mu \mathcal{S}'^{\mu}_{\alpha\beta} e^\alpha_a e^\beta_b \\
\kappa^{(2)}_{ab} &= \mathcal{L}_{\vec{T}_2}\kappa_{ab} - D_aD_bQ_2 - Q_2 n^\mu n^\nu R_{\alpha\mu\beta\nu} e^\alpha_a e^\beta_b + Q_2\kappa_{ac}\kappa^c_b - \ldots.
\end{aligned}
$$

where $K_{1\alpha\beta} = Y' n_\alpha n_\beta + n_\alpha \tau'_\beta + n_\beta \tau'_\alpha + K_1{}^t{}_{\alpha\beta}$

Second order perturbed matching conditions

Take the **first order matched configuration**.

Ingredients: to compute the 2nd order perturbations of the first and second fund. forms: $q^{(2)} \equiv \partial^2_\varepsilon q_\varepsilon|_{\varepsilon=0}$ and $\kappa^{(2)} \equiv \partial^2_\varepsilon \kappa_\varepsilon|_{\varepsilon=0}$:

- **Perturbed metric tensor** K_2
- **1st order deformation vector** of Σ (unknown): $\vec{Z}_2 \to Q_2, \vec{T}_2$

Theorem (Mars (2005)): perturbations of hypersurfs.

$$
\begin{aligned}
q^{(2)}_{ab} &= \mathcal{L}_{\vec{T}_2} q_{ab} + 2Q_2\kappa_{ab} + K_{2\alpha\beta} e^\alpha_a e^\beta_b + 2\mathcal{L}_{\vec{T}_1} q^{(1)}_{ab} - \mathcal{L}_{\vec{T}_1}\mathcal{L}_{\vec{T}_1} q_{ab} + \\
&+ \mathcal{L}_{2Q_1\vec{\tau}' - 2Q_1\kappa(\vec{T}_1) - D_{\vec{T}_1}\vec{T}_1} q_{ab} + 2D_aQ_1D_bQ_1 \\
&+ 2\left(T_1{}^cT_1{}^d\kappa_{cd} - 2\vec{T}_1(Q_1) + 2Q_1Y'\right)\kappa_{ab} + \\
&+ 2Q_1^2\left(-n^\mu n^\nu R_{\alpha\mu\beta\nu} e^\alpha_a e^\beta_b + \kappa_{ac}\kappa^c_b\right) - 4Q_1 n_\mu \mathcal{S}'^\mu{}_{\alpha\beta} e^\alpha_a e^\beta_b \\
\kappa^{(2)}_{ab} &= \mathcal{L}_{\vec{T}_2}\kappa_{ab} - D_aD_bQ_2 - Q_2 n^\mu n^\nu R_{\alpha\mu\beta\nu} e^\alpha_a e^\beta_b + Q_2\kappa_{ac}\kappa^c_b - \ldots .
\end{aligned}
$$

Theorem (Mars (2005)): 2nd order matching conditions are fulfilled:

iff $\exists\, Q_2^\pm$ and $\vec{T}_2^\pm$ such that $q^{(2)+}_{ab} = q^{(2)-}_{ab}, \quad \kappa^{(2)+}_{ab} = \kappa^{(2)-}_{ab}$

Perturbed matching

Perturbed matching conditions to second order:

$$q^{(1)+}_{ab} = q^{(1)-}_{ab}, \quad \kappa^{(1)+}_{ab} = \kappa^{(1)-}_{ab}, \quad q^{(2)+}_{ab} = q^{(2)-}_{ab}, \quad \kappa^{(2)+}_{ab} = \kappa^{(2)-}_{ab}$$

- $q^{(1)}_{ab}$, $\kappa^{(1)}_{ab}$, $q^{(2)}_{ab}$, $\kappa^{(2)}_{ab}$ are **gauge invariant** under **spacetime** perturbation gauge transformations **by construction**. But they are not hypersurface-gauge invariant.
- However, *the equations* are **gauge invariant** under both **spacetime** and **hypersurface** perturbation gauge transformations
- Fulfilling the matching conditions at each order requires *showing the existence* of two vectors $\vec{Z}^{\pm}$ (at each order) such that these equations are satisfied
- $\vec{Z}^{\pm}$ are **gauge dependent** (both spacetime and hypersurface). Both $(\pm)$ can be set to zero simultaneously using spacetime gauges. But one has to be careful, then.
- A **hypersurface gauge** can be used to set either T^{+} or T^{-} to zero, but not both.

Revisiting Hartle's model

Geometric approach

1) Build a static and spher. symm. background configuration

2) Add stationary and axisymm. metric and hypersurface perturbations

3) Perturbed matching

- **3.1)** 1st order
- **3.2)** 2nd order

Model of isolated rotating star in equilibrium

4) Matter content

5) Particularize the previous matching conditions

Revisiting Hartle's model

Geometric approach

1) Build a static and spher. symm. background configuration

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Revisiting Hartle's model

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1) Build a static and spher. symm. background configuration ✓

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- **3.1)** First order
- **3.2)** Second order

Model of isolated rotating star in equilibrium

1. Matter content
2. Particularize the previous matching conditions

Geometric approach

Family of metrics

$$
\begin{aligned}
g_\varepsilon &= -e^{\nu(r)}\left(1+2\varepsilon^2 h(r,\theta)\right)dt^2 + e^{\lambda(r)}\left(1+2\varepsilon^2 m(r,\theta)\right)dr^2 \\
&\quad + r^2(1+2\varepsilon^2 k(r,\theta))\left[d\theta^2+\sin^2\theta(d\varphi-\varepsilon\omega(r,\theta)dt)^2\right] + O(\varepsilon^3).
\end{aligned}
$$

Perturbation tensors: Take $\varepsilon-$derivatives in $\varepsilon = 0$

$$
\begin{aligned}
K_1 &= \partial_\varepsilon g_\varepsilon|_{\varepsilon=0}, \\
K_2 &= \partial^2_\varepsilon g_\varepsilon|_{\varepsilon=0}
\end{aligned}
$$

Introduce axisymmetric **deformation vectors** (unknowns)

$$
\begin{aligned}
\vec{Z}_1^{\pm} &= Q_1^{\pm}(\tau,\vartheta)\vec{n} + T_1^{t\,\pm}(\tau,\vartheta)\vec{e}_1 + T_1^{\varphi\,\pm}(\tau,\vartheta)\vec{e}_2 + T_1^{\theta\,\pm}(\tau,\vartheta)\vec{e}_3 \\
\vec{Z}_2^{\pm} &= Q_2^{\pm}(\tau,\vartheta)\vec{n} + T_2^{t\,\pm}(\tau,\vartheta)\vec{e}_1 + T_2^{\varphi\,\pm}(\tau,\vartheta)\vec{e}_2 + T_2^{\theta\,\pm}(\tau,\vartheta)\vec{e}_3
\end{aligned}
$$

Geometric approach

Family of metrics

$$
\begin{aligned}
g_\varepsilon &= -e^{\nu(r)}\left(1+2\varepsilon^2 h(r,\theta)\right)dt^2 + e^{\lambda(r)}\left(1+2\varepsilon^2 m(r,\theta)\right)dr^2 \\
&\quad + r^2(1+2\varepsilon^2 k(r,\theta))\left[d\theta^2+\sin^2\theta(d\varphi-\varepsilon\omega(r,\theta)dt)^2\right] + O(\varepsilon^3).
\end{aligned}
$$

Perturbation tensors: Take $\varepsilon-$derivatives in $\varepsilon = 0$

$$
\begin{aligned}
K_1 &= -2r^2\sin^2\theta\omega dt d\varphi \\
K_2 &= \left(-4e^{\nu}h(r,\theta)+2r^2\sin^2\theta\omega^2(r,\theta)\right)dt^2 + 4e^{\lambda}m(r,\theta)dr^2 \\
&\quad + 4r^2 k(r,\theta)d\Omega^2
\end{aligned}
$$

Introduce axisymmetric **deformation vectors** (unknowns)

$$
\begin{aligned}
\vec{Z}_1^{\pm} &= Q_1^{\pm}(\tau,\vartheta)\vec{n} + T_1^{t\,\pm}(\tau,\vartheta)\vec{e}_1 + T_1^{\varphi\,\pm}(\tau,\vartheta)\vec{e}_2 + T_1^{\theta\,\pm}(\tau,\vartheta)\vec{e}_3 \\
\vec{Z}_2^{\pm} &= Q_2^{\pm}(\tau,\vartheta)\vec{n} + T_2^{t\,\pm}(\tau,\vartheta)\vec{e}_1 + T_2^{\varphi\,\pm}(\tau,\vartheta)\vec{e}_2 + T_2^{\theta\,\pm}(\tau,\vartheta)\vec{e}_3
\end{aligned}
$$

Revisiting Hartle's model

Geometric approach

1) Build a static and spher. symm. background configuration ✓

2) Add stationary and axisymm. metric and hypersurface perturbations ✓

3) Perturbed matching

- **3.1)** First order
- **3.2)** Second order

Model of isolated rotating star in equilibrium

1. Matter content
2. Particularize the previous matching conditions

First order perturbations

Building the 1st order perturbation of the 1st and 2nd fundamental forms

- Background (already matched)
  - Metric: $g_0^{\pm}$
  - Unit normal $\vec{n}^{\pm}$
- First order
  - Hypersurface-deformation vector $\vec{Z}_1$ (unknown) $\rightarrow Q_1, \vec{T}_1$
  - Metric-perturbation tensor K_1

Perturbed first fundamental form

$$
\begin{aligned}
q_{ab}^{(1)} \quad = \quad & 2e^{\nu}\left(\frac{\nu_{,r}}{2}e^{-\lambda/2}Q_1 - T_1^t{}_{,\tau}\right)d\tau^2 + 2r_0^2\sin^2\vartheta(T_1^{\varphi}{}_{,\tau} - \omega)d\tau d\phi \\
& + 2\left(r_0^2 T_1^{\theta}{}_{,\tau} - e^{\nu}T_1^t{}_{,\vartheta}\right)d\tau d\vartheta + r_0^2\sin 2\vartheta T_1^{\theta}d\phi^2 + 2r_0^2\sin^2\vartheta T_1^{\varphi}{}_{,\vartheta}\, d\phi d\vartheta \\
& + 2r_0(-e^{-\lambda/2}Q_1 + r_0 T_1^{\theta}{}_{,\vartheta})d\vartheta^2
\end{aligned}
$$

First order perturbations

Building the 1st order perturbation of the 1st and 2nd fundamental forms

- Background (already matched)
  - Metric: $g_0^{\pm}$
  - Unit normal $\vec{n}^{\pm}$
- First order
  - Hypersurface-deformation vector $\vec{Z}_1$ (unknown) $\rightarrow Q_1, \vec{T}_1$
  - Metric-perturbation tensor K_1

Perturbed second fundamental form

$$
\begin{aligned}
\kappa_{ab}^{(1)} \quad = \quad & \left(\frac{1}{4} e^{\nu-\lambda} \left(Q_1 \left(\lambda,_r \nu,_r - 2 \left(\nu,_{rr} + \nu,_r^2 \right) \right) + 4 e^{\frac{\lambda}{2}} \nu,_r T_1^t,_\tau \right) - Q_1,_{\tau\tau} \right) d\tau^2 \\
& + 2 r_0 e^{-\lambda/2} \sin^2 \vartheta (\omega - T_1^\varphi,_\tau + r_0 \omega,_r) d\tau d\phi \\
& + \left(e^{-\frac{\lambda}{2}} \left(e^{\nu} \nu,_r T_1^t,_\vartheta - 2 r_0 T_1^\theta,_\tau \right) - 2 Q_1,_{\tau,\vartheta} \right) d\tau d\vartheta \\
& - \sin \vartheta e^{-\lambda} \left(\cos \vartheta e^{\lambda} Q_1,_\vartheta + \sin \vartheta \left(\frac{r_0 \lambda,_r}{2} - 1 \right) Q_1 + 2 r_0 \cos \vartheta e^{\frac{\lambda}{2}} T_1^\theta \right) d\phi^2 \\
& - 2 r_0 e^{-\lambda/2} \sin^2 \vartheta T_1^\varphi,_\vartheta d\phi d\vartheta + \left\{ - Q_1,_{\vartheta\vartheta} - \frac{1}{2} e^{-\lambda} \left(r_0 \lambda,_r - 2 \right) Q_1 - 2 r_0 e^{-\frac{\lambda}{2}} T_1^\theta,_\vartheta \right\} d\vartheta^2
\end{aligned}
$$

First order perturbations

- Theorem 1 in [Mars2005]: Find $\vec{Z}^{\pm}$ that solve the system

$$[q'_{ab}] = 0\ , \quad [\kappa'_{ab}] = 0.$$

- Integrate it to determine $[\omega]$, $[\omega_{,r}]$, $[T_1]$ and $Q_1^{\pm}$
- Results in

$$\begin{aligned}
&[\omega] = b_1, \ \ [\omega_{,r}] = 0, \\
&[Q_1] = 0, \ \ Q_1^{+}[\lambda_{,r}] = 0, \\
&[T_1^t] = C_1, \ \ [T_1^{\varphi}] = b_1\tau + C_2, \ \ [T_1^{\theta}] = 0.
\end{aligned}$$

C_1 and C_2 cannot be determined due to the isometries of the background *(M.Mars, F.C.Mena, R.Vera (2007))*

Revisiting Hartle's model

Geometric approach

1) Build a static and spher. symm. background configuration ✓

2) Add stationary and axisymm. metric and hypersurface perturbations ✓

3) Perturbed matching

- **3.1)** First order ✓
- **3.2)** Second order

Model of isolated rotating star in equilibrium

1. Matter content
2. Particularize the previous matching conditions

Second order perturbations

Building the perturbation of the first and second fundamental forms

- Background (matched)
  - Metric: $g_0^{\pm}$
  - Tangent basis and unit normal $\{\vec{e}_i^{\pm}\}$, $\vec{n}^{\pm}$
- First order (matched)
  - Hypersurface deformation vector $\vec{Z}_1$
  - Metric-perturbation tensor K_1
- Second order
  - Hypersurface deformation vector $\vec{Z}_2$ (unknown)
  - Metric-perturbation tensor K_2

$$
\begin{aligned}
q_{ab}^{(2)} &= ...(really\ long\ expression)..., \\
\kappa_{ab}^{(2)} &= ...(much\ longer)...
\end{aligned}
$$

Second order perturbations

Theorem 1 in [Mars2005]. Find $\vec{Z}_2^{\pm}$ that solve the system

$$[h''_{ab}] = 0\ , \quad [\kappa''_{ab}] = 0.$$

Results (2nd order integration constants in blue)

$$\begin{aligned}
[T_2^t] &= -H_0\tau + H_1, \\
[T_2^\varphi] &= 2b_1(T_1^t + \tau T_1^\theta \cot\vartheta) + D_2, \\
\left[T_2^\theta\right] &= (b_1\tau\cos\vartheta(b_1\tau - 2T_1^\varphi) - F_0)\sin\vartheta, \\
\left[\tilde{Q}_2\right] &= q\cos\vartheta + \mathcal{Q}, \\
[h] - \left[\frac{\nu_{,r}\, e^{-\lambda/2}\tilde{Q}_2}{4}\right] &= \frac{H_0}{2}, \\
[h_{,r}] - \left[\left(\frac{\nu_{,r}\, e^{-\lambda/2}}{4}\right)_{,r}\tilde{Q}_2\right] &= \frac{\nu_{,r}}{2}[m], \\
[k] - \frac{[e^{-\lambda/2}\tilde{Q}_2]}{2r_0} &= \frac{F_0}{2}\cos\vartheta, \\
[k_{,r}] - \left[\left(\frac{e^{-\lambda/2}}{2r}\right)_{,r}\tilde{Q}_2\right] &= \frac{e^{\lambda/2} q\cos\vartheta}{2r_0^2} + \frac{[m]}{r_0},
\end{aligned}$$

Revisiting Hartle's model

Geometric approach

1) Build a static and spher. symm. background configuration ✓

2) Add stationary and axisymm. metric and hypersurface perturbations ✓

3) Perturbed matching

- **3.1)** First order ✓
- **3.2)** Second order ✓

Model of isolated rotating star in equilibrium

1. Matter content
2. Particularize the previous matching conditions

Field equations

- KVFs: timelike $\vec{\xi}^{\pm}$ and (unique) axial $\vec{\eta}$
- We consider a perturbed perfect fluid: Energy momentum tensor of the ε-family $T^{\varepsilon}_{\alpha\beta} = (E^{\varepsilon} + P^{\varepsilon})u^{\varepsilon}_{\alpha}u^{\varepsilon}_{\beta} + P^{\varepsilon}g^{\varepsilon}_{\alpha\beta}$ with $P_{\varepsilon} = P_{\varepsilon}(E_{\varepsilon})$
- 4-velocity : $g_{\varepsilon}(\vec{u}_{\varepsilon}, \vec{u}_{\varepsilon}) = -1$ and $\vec{u} \propto \vec{\xi} + \varepsilon\Omega\vec{\eta}$
- Also expand $\vec{u}_{\varepsilon} = \vec{u} + \varepsilon\vec{u}^{(1)} + \frac{1}{2}\varepsilon^2\vec{u}^{(2)} + O(\varepsilon^3)$ and

$$
\begin{aligned}
E^{\varepsilon} &= E + \varepsilon E^{(1)} + \frac{\varepsilon^2}{2}E^{(2)} + O(\varepsilon^3) \\
P^{\varepsilon} &= P + \varepsilon P^{(1)} + \frac{\varepsilon^2}{2}P^{(2)} + O(\varepsilon^3)
\end{aligned}
$$

Second order field equations

In progress: show that the second order functions must finally be of the form $h(r,\theta) = h_0(r) + h_2(r)P_2(\cos\theta)$ and the same for m and k, given the two problems linked by the matching conditions found.

$l = 0$ **sector:**

Perfect fluid

- Perturbed 4-velocity $\vec{u}^{(2)} \propto \partial_t$
- Define the pressure perturbation factor (following Hartle): $\tilde{\mathcal{P}}_0 := P_0^{(2)}/(2(E+P))$.
- 1st order ODE system for $\{m_0^+, \tilde{\mathcal{P}}_0\}$ and algebraic equation for $\tilde{h}_0^+$.
- BC on $\{m_0^+, \tilde{\mathcal{P}}_0\}$ so that central density is fixed.

Asymptotically flat vacuum

- Solutions of the EFE's

$$h_0^-(r_-) = -\frac{1}{r_- - 2M}\left(\delta M - \frac{J^2}{r_-^3}\right),$$

$$r_- e^{-\lambda_-} m_0^-(r_-) = \delta M - \frac{J^2}{r_-^3}.$$

Second order field equations

In progress: show that the second order functions must finally be of the form $h(r,\theta) = h_0(r) + h_2(r)P_2(\cos\theta)$ and the same for m and k, given the two problems linked by the matching conditions found.

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- 1st order ODE system for $\{m_0^+, \tilde{\mathcal{P}}_0\}$ and algebraic equation for $\tilde{h}_0^+$.
- BC on $\{m_0^+, \tilde{\mathcal{P}}_0\}$ so that central density is fixed.

Asymptotically flat vacuum

- Solutions of the EFE's

$$h_0^-(r_-) = -\frac{1}{r_- - 2M}\left(\delta M - \frac{J^2}{r_-^3}\right),$$

$$r_- e^{-\lambda_-} m_0^-(r_-) = \delta M - \frac{J^2}{r_-^3}.$$

Second order matching in the $l = 0$ sector

$\{[q^{(2)}_{ab}] = 0, [\kappa^{(2)}_{ab}] = 0\}$ and field equations for $l = 0$ imply:

For the metric functions

$$[h_0] = \frac{H_0}{2}\,,\; [h_0'] = \frac{a - M}{a(a-2M)}[m_0]\,,\; [m_0] = -4\pi\frac{a^3}{M}[E]\tilde{\mathcal{P}}_0(a)$$

The matching condition on m_0 determines the excess of mass δM in terms of interior quantities. In terms of Hartle's functions and notation:

$$[m_0^H] = -4\pi\frac{a^3}{M}(a-2M)E(a)p_0^{H*}(a)$$

$$\delta M = m_0^H(a) + \frac{J^2}{a^3} + 4\pi\frac{a^3}{M}(a-2M)E(a)p_0^{H*}(a).$$