Prime ideals in quantum matrices and Poisson-prime ideals in the semi-classical limit

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13th June 2015

Background

Throughout: k is a field, $k = \overline{k}$, $char(k) = 0$.

Quantum algebra A

Non-commutative k-algebra with parameter $q \in k^{\times}$, $q^n \neq 1$.

Poisson algebra R

Commutative k-algebra with Poisson bracket.

Informally, Poisson bracket induced from non-commutative multiplication by letting $q \to 1$.

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Aim

Understand the topological structure of the spaces of prime ideals:

$$spec(A) = \{\text{all prime ideals in } A\},$$

$$pspec(R) = \{\text{all Poisson-prime ideals in } R\}.$$

Stratification

In a quantum algebra A:

- $\mathcal{H} = (k^\times)^r$ acting rationally on A by k-algebra automorphisms.
- $\mathcal{H}$-primes: the prime ideals invariant under the action of $\mathcal{H}$.
- The stratum associated to a $\mathcal{H}$-prime J in A is

$$spec_J(A) = \{P \in spec(A) : \bigcap_{h \in \mathcal{H}} h(P) = J\}.$$

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- Goodearl-Letzter stratification: study $spec_J(A)$ using the algebra

$$A_J := A/J[\mathcal{E}_J^{-1}], \quad \mathcal{E}_J = \{\text{non-zero eigenvectors in } A/J\}.$$

- There are homeomorphisms

$$spec_J(A) \approx spec(A_J) \approx spec(Z(A_J)).$$

- $Z(A_J)$ is a commutative Laurent polynomial ring.

Stratification

In a Poisson algebra R:

- $\mathcal{H} = (k^\times)^r$ acting rationally on R by Poisson algebra automorphisms.
- $\mathcal{H}$-primes: the Poisson-prime ideals invariant under the action of $\mathcal{H}$.
- The stratum associated to a $\mathcal{H}$-prime J in R is

$$pspec_J(R) = \{P \in pspec(R) : \bigcap_{h \in \mathcal{H}} h(P) = J\}.$$

- Goodearl-Letzter stratification: study $pspec_J(R)$ using the algebra

$$R_J := R/J[\mathcal{E}_J^{-1}], \quad \mathcal{E}_J = \{\text{non-zero eigenvectors in } R/J\}.$$

- There are homeomorphisms

$$pspec_J(R) \approx pspec(R_J) \approx spec(PZ(R_J)).$$

- $PZ(R_J)$ is a commutative Laurent polynomial ring.

Inclusions between primes

Stratification tells us all about the individual strata, but very little about interactions between primes from different strata.

A bijection $\psi : \mathit{spec}(A) \longrightarrow \mathit{pspec}(R)$ is a homeomorphism if and only if ψ, ψ^{-1} preserve inclusions of primes.

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Idea (Brown, Goodearl)

Capture information about inclusions of primes in different strata by studying the algebras

$$Z\Big(A/J\big[\mathcal{E}_{JK}^{-1}\big]\Big),$$

$$\mathcal{E}_{JK} = \{\text{non-zero eigenvectors of } A/J \text{ which are not in } K/J\},$$

for each comparable pair of $\mathcal{H}$-primes $J \subset K$.

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Idea (Brown, Goodearl)

Capture information about inclusions of Poisson-primes in different strata by studying the algebras

$$PZ\left(R/J[\mathcal{E}_{JK}^{-1}]\right),$$

$$\mathcal{E}_{JK} = \{\text{non-zero eigenvectors of } R/J \text{ which are not in } K/J\},$$

for each comparable pair of $\mathcal{H}$-primes $J \subset K$.

Quantum Matrices and their Semi-classical Limit

$\mathcal{O}_q(M_{m,n})$

Commutation relations: for each 2×2 submatrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$ab = qba \quad ac = qca$$
$$bd = qdb \quad cd = qdc$$
$$bc = cb$$
$$ad - da = (q - q^{-1})bc$$

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Commutative multiplication; Poisson bracket defined on each 2×2 submatrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

$$\{a, b\} = ab \quad \{a, c\} = ac$$
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When $m = n$, define

$$\mathcal{O}_q(SL_n) = \mathcal{O}_q(M_n)/(D_q - 1), \qquad \mathcal{O}(SL_n) = \mathcal{O}(M_n)/(D - 1).$$

Step 1: find the sets E_{JK}

Recall

Let $J \subset K$ be $\mathcal{H}$-primes in A or R. We are interested in the algebras

$$Z_{JK} := Z\Big(A/J[\mathcal{E}_{JK}^{-1}]\Big) \quad \text{and} \quad PZ_{JK} := PZ\Big(R/J[\mathcal{E}_{JK}^{-1}]\Big),$$

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In fact, it is sufficient to find multiplicative (Ore) sets $E_{JK} \subset \mathcal{E}_{JK}$ satisfying:

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For $\mathcal{O}_q(SL_3)$ and $\mathcal{O}(SL_3)$, can take the generators of E_{JK} to be the lacunary sequences starting at the white boxes of the Cauchon diagram of K.

Step 2: compute the centres

Theorem

Let $J \subset K$ be $\mathcal{H}$-primes in $\mathcal{O}_q(SL_3)$, $\mathcal{O}(SL_3)$. Then Z_{JK} and PZ_{JK} are mixed polynomial/Laurent polynomial rings, and we can find a presentation

$$Z_{JK} = k[Z_1^{\pm 1}, \ldots, Z_s^{\pm 1}, Z_{s+1}, \ldots, Z_t],$$
$$PZ_{JK} = k[z_1^{\pm 1}, \ldots, z_s^{\pm 1}, z_{s+1}, \ldots, z_t],$$

such that the Z_i are products of quantum minors, the z_i are products of minors, and z_i is obtained from Z_i by replacing each quantum minor in Z_i with the corresponding minor.

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The proof is by direct computation: just about possible for SL_3, definitely not recommended for larger algebras.

Homeomorphism

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Theorem

Let k be an algebraically closed field of characteristic zero, and $q \in k^\times$ not a root of unity. Then there is a homeomorphism

$$spec(\mathcal{O}_q(SL_3)) \longrightarrow pspec(\mathcal{O}(SL_3)),$$

and hence also a homeomorphism

$$prim(\mathcal{O}_q(SL_3)) \longrightarrow pprim(\mathcal{O}(SL_3)).$$