Homogenization of functionals with linear growth in the context of $\mathcal{A}$-quasiconvexity.

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joint work with José Matias and Pedro M. Santos (IST)

(SISSA, Trieste, Italy)

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$\mathcal{A}$-quasiconvexity: A function $f : \mathbb{R}^d \to \mathbb{R}$ is quasiconvex at $\xi \in \mathbb{R}^d$ if

$$f(\xi) \leqslant \int_Q f(\xi + w(y))\, \mathrm{d}y, \quad w \in C^\infty_{Q-\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^d) : \mathcal{A}w = 0, \int_Q w = 0.$$

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$\mathcal{A} = \mathbf{curl}$ gives the most familiar case $F(u) = \int_\Omega f(x,u(x),\nabla u(x))\mathrm{d}x$, but now one can consider $\mathcal{A} = \mathbf{div}$, or other operators (see Maxwell's equations).

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$\mathcal{A}$-quasiconvexity is the condition on f that ensures lower semicontinuity of F.

Objectives

To find an integral representation formula for the functional

$$\mathcal{F}(\mu) = \inf_{\{u_n\}\subset L^1(\Omega;\mathbb{R}^d)} \left\{ \liminf_{n\to\infty} \int_\Omega f\left(\frac{x}{\varepsilon_n}, u_n(x)\right) \mathrm{d}x : \quad u_n \overset{*}{\rightharpoonup} \mu, \right.$$

$$\left. \mathcal{A}u_n \overset{W^{-1,q}}{\to} 0, |u_n| \overset{*}{\rightharpoonup} \Lambda \text{ with } \Lambda(\partial\Omega) = 0 \right\},$$

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The condition of Λ not charging $\partial\Omega$ is needed otherwise some *extension properties* must be required on $\mathcal{A}$: it prevents concentrations at the boundary.

Previous results on related problems

Braides-Fonseca-Leoni studied the homogenization problem when f has growth of order $p > 1$:

$$0 \leqslant f(x, v) \leqslant C(1 + |v|^p),$$

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Here, f_{hom} is computed by keeping into account the $\mathcal{A}$-quasi-convexity constraint.

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p-equi-integrability: let $\{u_n\} \subset L^p$. For any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $|A| < \delta$, then

$$\int_A |u_n(x)|^p \mathrm{d}x < \varepsilon.$$

SISSA

The mail result

Theorem

Given $\mu \in \mathcal{M}(\Omega;\mathbb{R}^d) \cap \mathbf{ker}\,\mathcal{A}$, $\mu = u^a\mathcal{L}^N + \mu^s$, *if* f *satisfies the growth conditions* $C_1|\xi| \leqslant f(x,\xi) \leqslant C_2(1+|\xi|)$, *the following integral representation holds* $\mathcal{F}(\mu) = \mathcal{F}_{\mathcal{A}-\mathrm{hom}}(\mu)$

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$$\mathcal{F}_{\mathcal{A}-\mathrm{hom}}(\mu) := \int_\Omega f_{\mathcal{A}-\mathrm{hom}}(u^a(x))\,\mathrm{d}x + \int_\Omega f^\infty_{\mathcal{A}-\mathrm{hom}}\left(\frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|}(x)\right)\mathrm{d}|\mu^s|(x),$$

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$$f_{\mathcal{A}-\mathrm{hom}}(b) := \inf_{R\in\mathbb{N}} \inf \left\{ \fint_{RQ} f(x, b + w(x))\,\mathrm{d}x : \right.$$

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$$f^\infty_{\mathcal{A}-\mathrm{hom}}(b) = \limsup_{t\to\infty} \frac{f_{\mathcal{A}-\mathrm{hom}}(tb)}{t}.$$

Sketch of the proof I: properties of $f_{\mathcal{A}-\mathrm{hom}}$

Assume f is Lipschitz continuous in the second variable with constant $L > 0$, then $f_{\mathcal{A}-\mathrm{hom}}$ is too.

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Invariance under translations: define

$$f^{\gamma}_{\mathcal{A}-\mathrm{hom}}(b) := \inf_{R\in\mathbb{N}} \inf \left\{ \fint_{RQ} f(x+\gamma, b+w(x))\,\mathrm{d}x : \right.$$
$$\left. w \in L^1_{Q-\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^d) \cap \ker\mathcal{A},\ \fint_{RQ} w = 0 \right\},$$

then $f^{\gamma}_{\mathcal{A}-\mathrm{hom}}(b) = f_{\mathcal{A}-\mathrm{hom}}(b)$.

Sketch of the proof II: localization

Fix a sequence $\{\varepsilon_n\}$ and consider the localized functional

$$\mathcal{F}_{\{\varepsilon_n\}}(\mu;A)=\inf_{\{u_n\}\subset L^1(\Omega;\mathbb{R}^d)}\left\{\liminf_{n\to\infty}\int_A f\left(\frac{x}{\varepsilon_n},u_n(x)\right)\mathrm{d}x:\quad u_n\overset{*}{\rightharpoonup}\mu,\right.$$

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By regularizing μ one immediately obtains the upper bound

$$\mathcal{F}_{\{\varepsilon_n\}}(\mu;A) \leqslant C(|A| + |\mu|(\overline{A})).$$

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Subadditivity for nested sets: for any $A \subset\subset B \subset\subset C \subset \Omega$,

$$\mathcal{F}_{\{\varepsilon_n^1\}}(\mu;C) \leqslant \mathcal{F}_{\{\varepsilon_n^1\}}(\mu;B) + \mathcal{F}_{\{\varepsilon_n^1\}}(\mu;C\setminus\overline{A}),$$

for a suitable $\{\varepsilon_n^1\} \subset \{\varepsilon_n\}$.

Sketch of the proof III: measure properties

Proposition

Given $\mu \in \mathcal{M}(\Omega;\mathbb{R}^d)$ *and* $\{\varepsilon_n\}$*, there exists* $\{\varepsilon_n^1\} \subset \{\varepsilon_n\}$*, a sequence* $u_n^1 \overset{*}{\rightharpoonup} \mu$ *such that* $\mathcal{A}u_n^1 \to 0$*, and a bounded Radon measure* Φ_μ *such that*

$$f\left(\frac{x}{\varepsilon_n^1}, u_n^1(x)\right) \mathrm{d}x \overset{*}{\rightharpoonup} \Phi_\mu.$$

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$$f\left(\frac{x}{\varepsilon_n^1}, u_n^1(x)\right) \mathrm{d}x \overset{*}{\rightharpoonup} \Phi_\mu.$$

Moreover, for every open set $A \subset\subset \Omega$ *such that* $\Phi_\mu(\partial A) = 0$*, there exists* $\{\varepsilon_n^2\} \subset \{\varepsilon_n^1\}$ *such that*

$$\mathcal{F}_{\{\varepsilon_n^2\}}(\mu; A) = \Phi_\mu(A).$$

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Start with $\mu = u\mathcal{L}^N$. Prove

$$\frac{\mathrm{d}\Phi_\mu}{\mathrm{d}\mathcal{L}^N}(x_0) \leqslant f_{\mathcal{A}-\mathrm{hom}}(u(x_0)), \qquad \text{for } \mathcal{L}^N\text{-a.e. } x_0 \in \Omega.$$

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For a general $\mu \in \mathcal{M}(\Omega;\mathbb{R}^d)$, proceed by convolution: $u_n := \rho_n * \mu$.

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Absolutely continuous part: let $x_0 \in \Omega$ be a Lebesgue point for $\mu = u\mathcal{L}^N + \mu^s$. Then

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Singular part: use the translation invariance of $f_{\mathcal{A}-\mathrm{hom}}$ and the notion of *tangent measures* to obtain

$$\frac{\mathrm{d}\Phi_\mu}{\mathrm{d}|\mu^s|}(x_0) \geqslant f^\infty_{\mathcal{A}-\mathrm{hom}}\left(\frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|}(x_0)\right).$$

Comments and further directions

Extensions: The operator $\mathcal{A}$ has the *extension property* if the condition $\mathcal{A}u = 0$ in Ω implies that $\mathcal{A}$ can be extended to $\tilde{\mathcal{A}}$ such that $\tilde{\mathcal{A}}u = 0$ on $\tilde{\Omega} \supset \Omega$.

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It would be interesting to relax the conditions on f and also to add an explicit dependence on x both on $\mathcal{A}$ (see Davoli) and on f (introducing different scales).

Thank you very much for your attention!