Displacing pre-Lagrangians in contact toric manifolds.

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based on a joint work with Aleksandra Marinković

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Displaceability in Symplectic Geometry

(M, ω) symplectic manifold
$\Phi\colon M \to M$ is called a **Hamiltonian diffeomorphism** if $\Phi = \Phi_1$ for some isotopy $\{\Phi_t\}$ with $\Phi_0 = Id$, $\frac{d}{dt}\Phi_t = X_t \circ \Phi_t$ and $\omega(X_t, \cdot) = dh_t(\cdot)$ for some $h_t\colon M \to \mathbb{R}$.

$L \overset{\iota}{\hookrightarrow} M$, $\iota^*\omega = 0$ Lagrangian submanifold

L is called **non-displaceable** if for any Hamiltonian diffeomorphism $\Phi\colon M \to M$ one has that $\Phi(L) \cap L \neq \emptyset$
Otherwise called **displaceable**.

Symplectic toric manifold

is a symplectic manifold (M^{2n}, ω) equipped with an effective Hamiltonian action of a torus T^n. Then there exists a T-invariant **momentum map** $\mu\colon M \to \mathfrak{t}^* \cong \mathbb{R}^n$, such that

$$\iota(\xi_M)\omega = d\,\langle \mu, \xi \rangle \quad \forall\, \xi \in \mathfrak{t},$$

where ξ_M is the vector field on M corresponding to $\xi \in \mathfrak{t}$.

<u>Atiyah, Guillemin-Sternberg</u>: If M is compact, then the image of Φ is a convex polytope, convex hull of the images of the fixed points.

<u>Delzant:</u>
$\{$Delzant polytopes in $\mathbb{R}^n\} \Leftrightarrow \{$cpct, symplectic toric $2n$-mfds$\}$

Torus orbits: $\mu^{-1}(\mathrm{pt})$ for $\mathrm{pt} \in \mu(M)$
If $\mathrm{pt} \in \mathrm{Int}\mu(M)$ then $\mu^{-1}(\mathrm{pt})$ is a Lagrangian

Displaceability of Lagrangian toric fibers in symplectic toric manifolds

1. Any compact connected symplectic toric manifold contains a non-displaceable Lagrangian toric fiber ([FOOO], [EP], [GW])
2. "Most" of the toric fibers are displaceable by McDuff's method of probes

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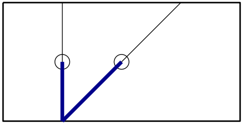

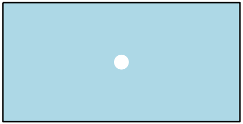

Contact manifolds and their Hamiltonian isotopies

(V^{2n+1}, ξ) co-oriented contact manifold,
Fix a 1-form α such that $\xi = \ker \alpha$ and $\alpha \wedge (d\alpha)^n \neq 0$.
Reeb vector field: v.field R_α uniquely defined by

$$d\alpha(R_\alpha, \cdot) = 0, \quad \alpha(R_\alpha) = 1.$$

Again: a time dependent function $h\colon V \times [0,1] \to \mathbb{R}$
$\rightsquigarrow$ Hamiltonian contact isotopy Φ_t obtained by integrating a time-dependent vector field X_t uniquely defiend by

$$\alpha(X_t) = h_t, \quad d\alpha(X_t, \cdot) = dh_t(R_\alpha)\alpha(\cdot) - dh_t(\cdot)$$

Note:
all contact isotopies starting at identity are Hamiltonian contact isotopies. (Recover h_t as $\alpha(X_t)$).

Pre-Lagrangians

A symplectization SV of (V, ξ) is

$$\{(p, \eta_p) \in T^*V \mid p \in V,\ \ker \eta_p = \xi_p,\ \text{same orientation}\} \cong V \times \mathbb{R}_+$$

with the symplectic structure induced from T^*V.
$\pi\colon SV \to V$ a natural projection

$L \subset V$ is a **pre-Lagrangian** if there exists a Lagrangian $\widetilde{L} \subset SV$ such that $\pi_{|\widetilde{L}}\colon \widetilde{L} \to L$ is a diffeomorphism.

Rmks:

- If V is a prequantization space, $S^1 \hookrightarrow V^{2n+1} \xrightarrow{p} W^{2n}$, and L' is a Lagrangian in W then $p^{-1}(L')$ is a pre-Lagrangian in V.
- If N is a Legendrian in V, i.e. $TN \subset \xi$, then

$$\bigcup_{p \in N} (\text{ Reeb orbit of } p) \text{ is a pre-Lagrangian in } V$$

Question:

Given a pre-Lagrangian L in V does there exist a contact isotopy Φ displacing it?

Contact toric manifolds

(V^{2n+1}, ξ) equipped with an effective action of a torus T^{n+1} by contact transformations. Generic torus orbits are pre-Lagrangians. Call them **pre-Lagrangian toric fibers**.

Any contact form α for ξ can be made T invariant by averaging. Each T-invariant contact form $\alpha \rightsquigarrow$ an α-**moment map** $\mu_\alpha \colon V \to \mathfrak{t}^*$ defined by $\mu_\alpha(p)(X) = \alpha_p(X_p)$

Contact moment map:

$$\mu \colon SV \to \mathfrak{t}^*, \quad \mu(p, \eta_p)(X) = \eta_p(X)$$

Moreover:

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & SV \\ & \searrow^{\mu_\alpha} & \downarrow{\mu} \\ & & \mathfrak{t}^* \end{array}$$

Examples

- $S^{2n-1} \subset \mathbb{C}^n$, $\xi = TS^{2n-1} \cap J(TS^{2n-1})$,
  torus T^n acts by rotating each copy of $\mathbb{C}$ with speed 1.
  Pre-Lagrangian toric fibers are
  $L = \{(z_1, \ldots, z_n) \in S^{2n-1}\,;\; |z_1|^2 = c_1^2, \ldots, |z_n|^2 = c_n^2\}$ for some $0 < c_1, \ldots, c_n < 1$, such that $\sum c_j^2 = 1$

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$$\mathbb{P}_+(T^*T^n) = T^n \times S^{n-1}$$

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  T^n acts by rotating the T^n factors, thus pre-Lagrangian toric fibers are $T^n \times \{\mathrm{pt}\}$, $\mathrm{pt} \in S^{2n-1}$.
- Given an integral symplectic toric manifold (M^{2n}, ω) one can define a toric action on its prequantization $p\colon V^{2n+1} \to M$. Toric fibers are $p^{-1}(\text{ Lagrangian toric fibers in } M)$.

Is it also true that

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- while "most" of them are displaceable?

NO for both!

- All pre-Lagrangian toric fibers in S^{2n-1}, $S^1 \times S^{2n}$, $n > 1$, are displaceable!
- All pre-Lagrangian toric fibers in $\mathbb{P}_+(T^*T^n) = T^n \times S^{n-1}$, $n > 1$ are non-displaceable!

Orderability

Eliashberg-Polterovich:
two contact isotopies $\{\Phi\}, \{\Psi\} \in \widetilde{Cont}_0(V, \xi)$ satisfy $\{\Phi\} \preceq \{\Psi\}$ if and only if $\{\Psi \circ \Phi^{-1}\}$ is generated by a non-negative Hamiltonian function.
This relation is always reflexive and transitive.
If it is also anti-symmetric then it defines a bi-invariant order on $\widetilde{Cont}_0(V, \xi)$ and the contact manifold (V, ξ) is called **orderable**.

Equivalently, a contact manifold is orderable if there are no contractible loops of contactomorphisms generated by a strictly positive contact Hamiltonian.

Quasimorphisms

$\lambda\colon (G,*) \to (\mathbb{R},+)$ is a **quasimorphism** if

$$\exists_{D\in\mathbb{R}}\ \forall_{g_1,g_2\in G}\ |\lambda(g_1 * g_2) - \lambda(g_1) - \lambda(g_2)| < D$$

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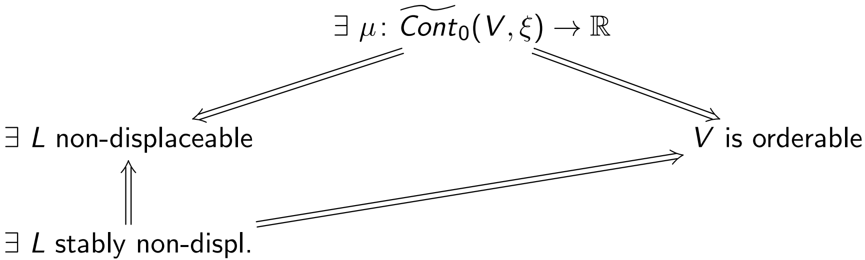
- is called **monotone** if $\{\Phi\} \preceq \{\Psi\} \Rightarrow \lambda(\{\Phi\}) \leq \lambda(\{\Psi\})$
- has **a vanishing property** if $U \subset V$, open and displaceable, i.e. there exist $\phi \in Cont_0(V,\xi)$ such that $\phi(U) \cap U = \emptyset$, then λ vanishes on all Hamiltonian isotopies $\{\Psi\} \in \widetilde{Cont}_0(V,\xi)$ with support in $[0,1] \times U$.



- (Borman-Zapolsky): existence of a monotone quasimorphism implies orderability (Eliashberg-Polterovich) and, if μ has a vanishing property, it also implies the existence of non-displaceable pre-Lagrangian torus (Entov-Polterovich),
- stable non-displaceability $\Rightarrow$ non-displaceability,
- (Eliashberg-Polterovich): existence of stably non-displaceable pre-Lagrangian implies orderability;

Compact contact toric manifolds that are NOT orderable:

- S^{2n-1}, $n \geq 2$,
- $S^1 \times S^{2n}$, $n \geq 2$,
  (orderability of $S^1 \times S^2$ is not known)
- more generally $T^k \times S^{2n+k-1}$, $k \geq 1$, $n \geq 2$;

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Eliashberg-Kim-Polterovich: For any Liouville manifold (M, ω) the ideal contact boundary of its n-stabilization is not orderable provided that $n \geq 2$.

Displaceability in S^{2n-1}

Pre-Lagrangian toric fibers are of the form

$$L = \{(z_1, \ldots, z_n) \in S^{2n-1}\, ;\ |z_1|^2 = c_1^2, \ldots, |z_n|^2 = c_n^2\}$$

for some $0 < c_1, \ldots, c_n < 1$, such that $\sum c_j^2 = 1$. The map

$$\tau_t(z_1, \ldots, z_n) = \frac{1}{cosh\, t + z_1 sinh\, t}(sinh\, t + z_1 cosh\, t, z_2, \ldots, z_n),$$

is a contactomorphism of S^{2n-1} for all $t \geq 0$. Each pre-Lagrangian L is displaced by τ_t for t big enough (Marinković-P.).

Also: complement of a point in S^{2n-1} is a Darboux ball, thus any non-trivial closed subset of the sphere is displaceable.

Methods for displacing

Rough idea:

- find some "well-understood" subset W, s.t. $L \subset W \subset V$,
- find $\phi \colon W \to W$ displacing L,
- extend ϕ to all of V;

How to find "well-understood" subsets?

Contact reduction

$$\begin{array}{ccc} L \subset \mu_G^{-1}(0) & \longrightarrow & V \\ \big\downarrow /G & & \\ L_0 \subset \mu_G^{-1}(0)/G =: V_0 & & \end{array}$$

Any $\phi \in Cont_0(V_0, \ker \alpha_0)$ can be lifted to $Cont_0(V, \ker \alpha)$.

If $L \subset \mu_G^{-1}(0) \subset V$ and $L_0 = L/G \subset V_0$ then

- L_0 displaceable in $V_0 \Rightarrow L$ displaceable in V,
- L non-displaceable in $V \Rightarrow L_0$ non-displaceable in V_0;

(This is a direct translation of a similar result of Abreu-Macarini in the symplectic setting)

Prequantization

Similarly, if $(V,\xi) \xrightarrow{p} (M,\omega)$ is a prequantization and L' a Lagrangian in M, then $L := p^{-1}(L')$ is a pre-Lagrangian in V and:

- L' displaceable in $M \Rightarrow L$ displaceable in V,
- L non-displaceable in $V \Rightarrow L'$ non-displaceable in M;

$$\begin{array}{ccc} S^1 & \longrightarrow & V \supset L \\ & & \Big\downarrow p \\ & & M \supset L' \end{array}$$

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In particular, for all $L' \subset M$ displaceable by McDuff's method of probes, $p^{-1}(L')$ is displaceable in V.

Contact cuts (Lerman)

- $(V, \xi = \ker \alpha)$ a contact manifold with S^1 action preserving α,
- $\mu_\alpha \colon V \to \mathbb{R}$ the corresponding moment map,
- S^1 acts freely on $\mu_\alpha^{-1}(0)$;

Then the **cut**

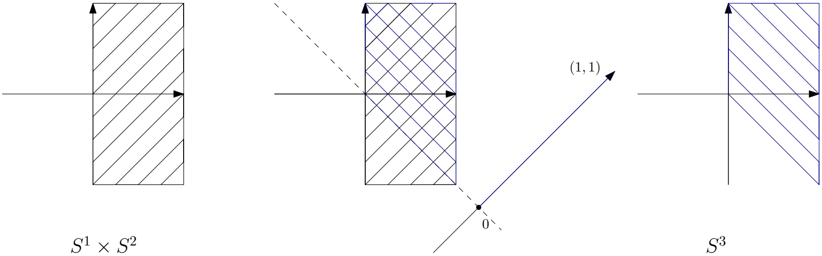
$$V_{[0,\infty)} := \{x \in V \mid \mu_\alpha(x) \geq 0\}/\sim,$$

where $x \sim x' \Leftrightarrow \mu_\alpha(x) = 0 = \mu_\alpha(x')$ and $x' = \lambda \cdot x$, some $\lambda \in S^1$, is naturally a contact manifold.
Moreover, the natural embedding of the reduced space $V_0 := \mu_\alpha^{-1}(0)/S^1$ into $V_{[0,\infty)}$ is contact and there is a contactomorphism

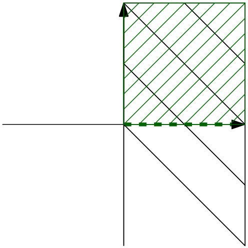
$$\left(V_{[0,\infty)} \setminus V_0\right) \cong \{x \in V \mid \mu_\alpha(x) > 0\}.$$

$\Rightarrow$ Any contact isotopy of the cut $V_{[0,\infty)}$ compactly supported in $\left(V_{[0,\infty)} \setminus V_0\right)$ can be extended to a contact isotopy of V.

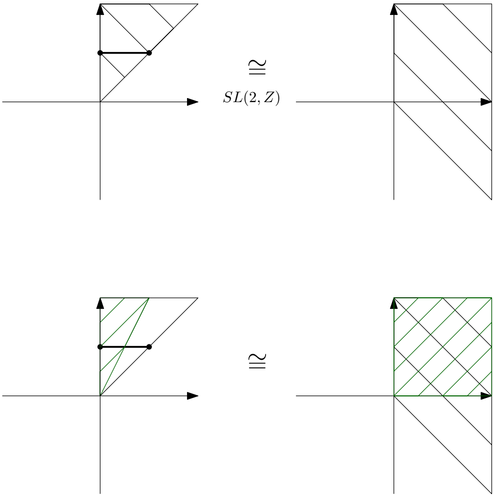
Example: T^2 toric action on $V = S^1 \times S^2$
(moment cone, $\mathbb{R} \times \mathbb{R}_{\geq 0}$, on the left)
Choose $S^1 = \{(t,t) \in T^2\}$. Then the moment cone for $V_{[0,\infty)}$ is as on the right picure and $V_{[0,\infty)}$ is contactomorphic to S^3 with the usual contact structure.



(S^3, ξ_{std}) is a prequantization of $\mathbb{CP}^1$.
All pre-Lagrangian toric fibers in S^3 which map to green region can be displaced by isotopies which are the lifts of the "probes-isotopies".

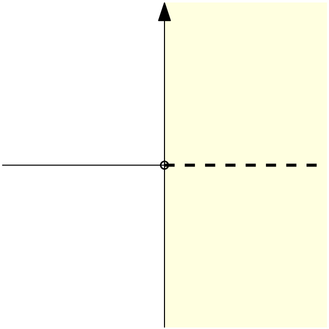


These isotopies are supported in $\left(V_{[0,\infty)} \setminus V_0\right)$ and thus can be extended to isotopies of $S^1 \times S^2$.
$\Rightarrow$ All pre-Lagrangian toric fibers in $S^1 \times S^2$ which map to green region are displaceable.


≅
$SL(2,Z)$
≅

Using this method we can displace

- all pre-Lagrangian toric fibers in $S^1 \times S^{2n}$, $n \geq 2$,
- all pre-Lagrangian toric fibers in $T^k \times S^{2n+k-1}$, $k \geq 1$, $n \geq 2$,
- most of pre-Lagrangian toric fibers in $S^1 \times S^2$:



(Non-)existence of a displaceable fiber

Example 1:

$\mathbb{P}_+(T^*T^n) = T^n \times S^{n-1}$ contact toric manifold,
pre-Lagrangian toric fibers are $T^n \times \{\mathrm{pt}\}$, $\mathrm{pt} \in S^{n-1}$.

ALL are non-displaceable:

Symplectization of $\mathbb{P}_+(T^*T^n)$ is $T^*T^n \setminus \{$ zero section $\}$,
Chaperon: Lagrangians $T^n \times \{\mathrm{pt}\}$ in T^*T^n are non-displaceable by contact isotopies

Note: the T^n action on $\mathbb{P}_+(T^*T^n)$ is free.

(Non-)existence of a displaceable fiber

Example 2:

$$(T^3 = S^1_{(t)} \times T^2_{(\theta_1,\theta_2)}, \xi_k = \ker(\cos(kt)\, d\theta_1 + \sin(kt)\, d\theta_2), \quad k > 1$$

with a free T^2 action rotating the T^2 component.

Pre-Lagrangian toric fibers are $\{t_0\} \times T^2$ and they are all non-displaceable.

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Pre-Lagrangian toric fibers are $\{t_0\} \times T^2$ and they are all non-displaceable.

(Isotopy Φ_1 displacing such a fiber could be used to construct a contactomorphism from $[-A, 0] \times T^2$ to $[-A, B] \times T^2$, $B > 0$, contradicting the classification of tight contact structures on $[a, b] \times T^2$.)

What about other contact toric manifolds with a free toric action?

Lerman's Classification:
Contact toric manifolds with a free toric action are:

- $(T^3 = S^1 \times T^2, \xi_k = \ker(\cos(kt)\, d\theta_1 + \sin(kt)\, d\theta_2), \ k > 1,$
- principal T^n bundles over S^{n-1}, (trivial for $n \neq 3$).

For $n = 3$ each such bundle is $T^2 \times S^3$ or $T^2 \times (S^3/\mathbb{Z}_p)$.
Moreover, it is contactomorphic to $(T^2 \times S^3, \ker \alpha)$, where α is

$$i(z_1\bar{z}_2 - \bar{z}_1 z_2)d\theta_1 + (z_1\bar{z}_2 + \bar{z}_1 z_2)d\theta_2 + \frac{i}{4}(z_1 d\bar{z}_1 - \bar{z}_1 dz_1 - (z_2 d\bar{z}_2 - \bar{z}_2 dz_2)),$$

or $T^2 \times (S^3/\mathbb{Z}_p)$ with the induced contact form (Marinković).
There we don't know if the toric fibers are non-displaceable.

Lerman's Classification $\Rightarrow$
contact toric manifolds for which the toric action is NOT free are:

- 3-dim lens spaces (include $S^1 \times S^2$),
- prequantizations of toric symplectic orbifolds,
- $T^k \times S^{2n-1-k}$;

All of them contain displaceable pre-Lagrangian toric fibers.

Guesses/Questions:

Could it be true that for a compact contact toric manifold

- all pre-Lagrangian toric fibers are displaceable IFF the manifold is not orderable,
- all pre-Lagrangian toric fibers are non-displaceable IFF the toric action is free?

Guesses/Questions:

Which contact toric manifolds contain a non-displaceable toric fiber and a displaceable ones, as symplectic toric manifolds do?

Definitely prequantizations of symplectic toric manifolds (these have displaceable fibers) that can be equipped with a monotone quasimorphism with a vanishing property.

Examples:
$\mathbb{RP}^{2n+1}$, prequantization of $\mathbb{CP}^n$, with a quasimorphism given by Givental's non-linear Maslov index.
Also, their quotients: lens spaces L_{2k}^{2n+1}.

Project in progress, with Yael Karshon and Sheila Sandon: extend Givental's construction of non-linear Maslov index to other lens spaces, and then to other prequantizations of toric symplectic manifolds.