The Ribes–Zalesskiĭ-Theorem revisited

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Notation throughout:

- A a finite alphabet
- F the free group on A

Theorem (Ribes–Zalesskiĭ 1993)

The set product $H_1 \cdots H_n$ of any finite number of finitely generated subgroups H_i of F is closed in the profinite topology of F.

Original motivation: Rhodes type-II conjecture

Generalization (Ribes–Zalesskiĭ 1993)

For every extension closed pseudovariety $\mathbf{H}$ of groups, the product $H_1 \cdots H_n$ is closed in the pro-$\mathbf{H}$ topology of F provided that the constituents H_i are pro-$\mathbf{H}$ closed.

I need two concepts:

1. tree-like profinite graph
2. **H**-extendible subgroup of F

ad 1:

- a *profinite* graph is a projective limit of finite graphs
- a *connected* profinite graph is one all of whose finite quotients are connected as abstract graphs

Definition

A (connected) profinite graph Γ is *tree-like* if any two vertices u, v of Γ admit a smallest (w.r.t. containment) connected subgraph $[u, v]$ of Γ containing u and v — the *geodesic* between u and v.

ad 2:

- every finitely generated subgroup H of F can be encoded by a uniquely determined connected finite directed A-labeled pointed graph $\mathscr{H}$ (Stallings graph of H)
- the letters of A induce partial injective mappings on the vertex set of $\mathscr{H}$
- a *completion* $\overline{\mathscr{H}}$ of $\mathscr{H}$ is a connected finite directed A-labeled over-graph of $\mathscr{H}$ such that all letters of A induce permutations on the vertex set of $\overline{\mathscr{H}}$
- the permutation group thereby generated by A is the *transition group* of $\overline{\mathscr{H}}$.

Definition (Margolis–Sapir–Weil 2001)

A finitely generated subgroup H of F is $\mathbf{H}$-extendible if the Stallings graph $\mathscr{H}$ admits a completion $\overline{\mathscr{H}}$ whose transition group is in $\mathbf{H}$.

Theorem (B. Steinberg, K. A. 2004)

Let $\mathbf{H}$ *be a pseudovariety of groups and let* $\widehat{F_{\mathbf{H}}}$ *be the pro-*$\mathbf{H}$*-completion of* F*. Then the following are equivalent:*

1. *the Cayley-graph of* $\widehat{F_{\mathbf{H}}}$ *is tree-like*
2. *the set product* H_1H_2 *of any two* $\mathbf{H}$*-extendible subgroups* H_i *of* F *is pro-*$\mathbf{H}$*-closed in* F
3. *for every* $n \geq 1$*, the set product* $H_1 \cdots H_n$ *of any* $\mathbf{H}$*-extendible subgroups* H_i *of* F *is pro-*$\mathbf{H}$*-closed in* F*.*

For $\mathbf{H}$ = the pseudovariety of all groups this gives immediately the Ribes-Zalesskiĭ-Theorem (similarly for $\mathbf{H}$ extension closed).

Open Problem

Can condition (2) of the Theorem be replaced by

- every $\mathbf{H}$-extendible subgroup H of F is pro-$\mathbf{H}$-closed in F?

The theorem is true for "pseudovariety" replaced with "formation":

Theorem

Let $\mathfrak{F}$ be a formation of groups and let $\widehat{F_{\mathfrak{F}}}$ be the pro-$\mathfrak{F}$-completion of F. Then the following are equivalent:

1. *the Cayley graph of $\widehat{F_{\mathfrak{F}}}$ is tree-like*
2. *the set product $H_1 H_2$ of any two $\mathfrak{F}$-extendible subgroups H_i of F is pro-$\mathfrak{F}$-closed in F*
3. *for every $n \geq 1$, the set product $H_1 \cdots H_n$ of any $\mathfrak{F}$-extendible subgroups H_i of F is pro-$\mathfrak{F}$-closed in F.*

Why formations?

- definition of pro-$\mathfrak{X}$ topology of F is natural for $\mathfrak{X}$ a formation
- Ballester-Bolinches / Pin / Soler-Escrivà
- new examples!!

- original proofs depend on heavy use of inverse monoids
- Cayley graph $\Gamma(\widehat{F_{\mathfrak{F}}})$ being tree-like is equivalent to the profinite inverse monoid $\widehat{M}(\widehat{F_{\mathfrak{F}}})$ being F-inverse
- one works with the finite quotients of $\widehat{M}(\widehat{F_{\mathfrak{F}}})$ — looks at "point-like pairs" and "liftable tuples" — and afterwards translates back to the Cayley graph $\Gamma(\widehat{F_{\mathfrak{F}}})$
- obscures what's really going on: the tight connection between the geometry of the graph $\Gamma(\widehat{F_{\mathfrak{F}}})$ and the pro-$\mathfrak{F}$-topology of F

let H be an $\mathfrak{F}$-extendible subgroup of F and $\mathscr{H}$ its Stallings graph

let $\overline{\mathscr{H}}$ be a completion of $\mathscr{H}$ with transition group $T \in \mathfrak{F}$

there is a canonical graph morphism $\Gamma(T) \twoheadrightarrow \overline{\mathscr{H}}$

there is a (canonical) subgraph $\mathscr{H}^T$ of the Cayley graph $\Gamma(T)$ such that $\mathscr{H}^T \twoheadrightarrow \mathscr{H}$

there is a (canonical) subgraph $\mathscr{H}^{\widehat{F_{\mathfrak{F}}}}$ of the Cayley graph $\Gamma(\widehat{F_{\mathfrak{F}}})$ such that $\mathscr{H}^{\widehat{F_{\mathfrak{F}}}} \twoheadrightarrow \mathscr{H}$

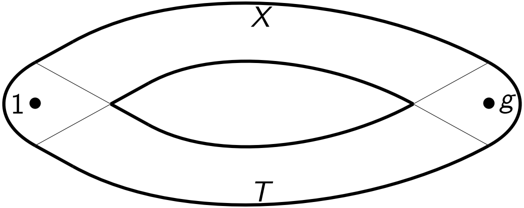
this leads to more transparent direct proofs

What forces a profinite group $\mathcal{G} = \langle A \rangle$ to have a tree-like Cayley graph?

Definition

Let G be an A-generated (pro)finite group. An *annular constellation* in G is a triple (X, g, T) where

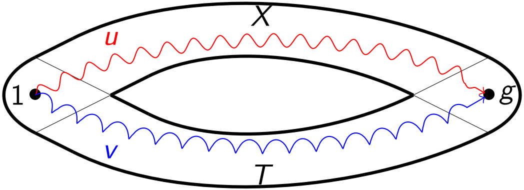
1. X, T are connected subgraphs of $\Gamma(G)$
2. $1, g$ are vertices of distinct connected components of $X \cap T$



Definition

Let G be a finite A-generated groups and (X, g, T) be an annular constellation of G. An A-generated co-extension $H \twoheadrightarrow G$ *dissolves* (X, g, T) if, for all $u, v \in F$ for which $[u]_G = g = [v]_G$ and $u : 1 \to g$ runs in X while $v : 1 \to g$ runs in T, the inequality $[u]_H \neq [v]_H$ holds.

That is, whenever in G we have:



then, in H we have $[u]_H \neq [v]_H$.

Theorem

An A-generated profinite group $\mathcal{G}$ has a tree-like Cayley graph if and only if each finite quotient G of $\mathcal{G}$ admits a finite quotient H of $\mathcal{G}$ which dissolves all annular constellations of G.

Given a group G, we want a co-extension $\widetilde{G} \twoheadrightarrow G$ which dissolves all annular constellations of G. This will be accomplished by this construction: let S be a finite simple group and R be a finitely generated free group; let $R(S)$ be the intersection of all normal subgroups N of R for which R/N is a direct power of S; $R(S)$ is a characteristic subgroup of R; if $S = C_p$ then $R(C_p) = R^p[R, R]$.

Definition

Let G be an A-generated finite group and let R be the kernel of the canonical morphism $F \twoheadrightarrow G$. The *A-universal S-extension* $G^{A,S}$ of G is defined by $G^{A,S} := F/R(S)$.

Theorem

$G^{A,S}$ dissolves all annular constellations of G.

Example

- let S be any finite simple non-abelian group
- let $\mathfrak{S}$ be the formation of all finite groups all of whose principal factors are isomorphic with S
- set $R_1 := F(S)$ and $R_{n+1} := R_n(S)$
- then $\widehat{F_{\mathfrak{S}}} = \varprojlim F/R_n$ has tree-like Cayley graph
- the set-product $H_1 \cdots H_n$ of any finite number n of pro-$\mathfrak{S}$-closed finitely generated subgroups H_i of F is pro-$\mathfrak{S}$ closed in F

Thanks!