FACULTADE DE MATEMÁTICAS

Universal Central Extensions of Lie–Rinehart Algebras

Xabier García Martínez

University of Santiago de Compostela

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- A K-Lie homomorphism and A-module homomorphism (called anchor map)

$$\alpha_L \colon L \to \mathtt{Der}_K(\mathrm{A})$$

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$$\alpha_L \colon L \to \mathtt{Der}_K(\mathrm{A})$$

- The following equality holds for every $a \in \mathrm{A}$ and $x, y \in L$.

$$[x, ay] = a[x, y] + x(a)y,$$

  where $x(a) = \alpha_L(x)(a)$.

Examples

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The set $\texttt{Der}_K(\mathrm{A})$ is a Lie K-algebra with Lie bracket $[D, D'] = DD' - D'D$, and anchor map the identity.

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Example

Lie–Algebroids.

Names

- Herz (1953): Pseudo–algèbre de Lie,
- Hochschild (1955): Regular restricted Lie algebra extension,
- Palais (1961): Lie d-ring,
- Rinehart (1963): (R, C) Lie algebra,
- de Barros (1964): (R, C)-espace d'Elie Cartan régulier et sans courbure,
- Bkouche (1966): (R, C) algèbre de Lie,
- Hermann (1967): Lie algebra with an associated module structure,
- Nelson (1967): Lie module,
- Kamber–Tondeur (1971): Sheaf of twisted Lie algebras,
- Illusie (1972): Algèbre de Lie sur C/R,
- Ieleman (1972): Lie algebra extension,
- Kastler–Stora (1985): Lie–Cartan pair,
- Beilinson–Schechtmann (1988): Atiyah algebra,
- Kostant–Sternberg (1990): $(A, \mathcal{C})$-system,
- Huebschmann (1990): Lie–Rinehart algebra,
- Kosmann–Schwarzbach–Magri (1990): Differential Lie algebra.

Changing A

$(f,\varphi)\colon (L,A) \to (L',A')$

$$\begin{array}{ccc} A\otimes L & \xrightarrow{(f,\varphi)} & A'\otimes L' \\ \downarrow & & \downarrow \\ L & \xrightarrow{\quad f \quad} & L' \end{array} \qquad \begin{array}{ccc} L\otimes A & \xrightarrow{(f,\varphi)} & L'\otimes A' \\ \downarrow & & \downarrow \\ A & \xrightarrow{\quad \varphi \quad} & A' \end{array}$$

The category of Lie–Rinehart Algebras

Changing A

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Keeping A

$$\begin{array}{ccc} L & \xrightarrow{f} & L' \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\alpha'} & \\ & \mathtt{Der}_K(\mathrm{A}) & \end{array}$$

Definition

Let L a Lie–Rinehart algebra over A. The *universal enveloping algebra* $\mathrm{U}_{\mathrm{A}} L$ is the K-algebra generated by the symbols $j(a), i(x)$ for each $a \in \mathrm{A}$ and $x \in L$ satisfying the relations

$$\begin{aligned} j(1) &= 1, \\ j(ab) &= j(a)j(b), \\ i(ax) &= j(a)i(x), \\ i([x, y]) &= i(x)i(y) - i(y)i(x), \\ i(x)j(a) &= j(a)i(x) + j\big(x(a)\big). \end{aligned}$$

Theorem (Rinehart, 1963)

If L is free as an A-module with basis $\{x_i, i \in I\}$, then the set

$$\{x_{i_1}^{s_1} x_{i_2}^{s_2} \cdots x_{i_r}^{s_r} : i_1 < i_2 < \cdots < i_r \text{ and } s_i \geq 0\},$$

form an A-basis of $\mathrm{U}_{\mathrm{A}}\, L$.

Universal Enveloping Algebra of Lie–Rinehart Algebras

Theorem (Rinehart, 1963)

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form an A-basis of $\mathrm{U}_{\mathrm{A}}\, L$.

Theorem (Kowalzig, 2009)

$\mathrm{U}_{\mathrm{A}}\, L$ has a left Hopf algebroid structure.

Left (A, L)-Module

Definition

A *left Lie–Rinehart (A, L)-module* over a Lie–Rinehart A-algebra L is a K-module $\mathcal{M}$ together with two operations

$$L \otimes \mathcal{M} \to \mathcal{M}, \qquad (x, m) \mapsto xm, \qquad \mathrm{A} \otimes \mathcal{M} \to \mathcal{M}, \qquad (a, m) \mapsto am,$$

such that the first one makes $\mathcal{M}$ into an L-module, the second map makes $\mathcal{M}$ into an A-module and additionally the compatibility conditions:

$$\begin{aligned}(ax)(m) &= a(xm),\\ x(am) &= a(xm) + x(a)m, \qquad a \in \mathrm{A}, m \in \mathcal{M} \text{ and } x \in L.\end{aligned}$$

Right (A, L)-Module

Definition

A *right Lie–Rinehart* (A, L)*-module* over a Lie–Rinehart A-algebra L is a K-module $\mathcal{M}$ together with two operations

$$\mathcal{M} \otimes L \to \mathcal{M}, \qquad (m, x) \mapsto mx, \qquad \mathrm{A} \otimes \mathcal{M} \to \mathcal{M}, \qquad (a, m) \mapsto am,$$

such that the first one makes $\mathcal{M}$ into an L-module, the second map makes $\mathcal{M}$ into an A-module and additionally the compatibility conditions:

$$\begin{aligned} m(ax) &= (am)x, \\ m(ax) &= a(mx) - x(a)m, \qquad a \in \mathrm{A}, m \in \mathcal{M} \text{ and } x \in L. \end{aligned}$$

Homology

Definition

Let $\mathcal{M}$ a right (A, L)-module. Consider the chain complex:

$$\partial\colon \mathcal{M} \otimes_{\mathrm{A}} \Lambda^n_{\mathrm{A}}(L) \to \mathcal{M} \otimes_{\mathrm{A}} \Lambda^{n-1}_{\mathrm{A}}(L),$$

$$\begin{aligned}\partial\big(m \otimes_{\mathrm{A}} (x_1, \dots, x_n)\big) = {} & \sum_{i=1}^{n} (-1)^{(i-1)}\, m x_i \otimes_{\mathrm{A}} (x_1, \dots, \hat{x}_i, \dots, x_n) \\ & + \sum_{j<k} (-1)^{j+k}\, m \otimes_{\mathrm{A}} ([x_j, x_k], x_1, \dots, \hat{x}_j, \dots, \hat{x}_k, \dots, x_n),\end{aligned}$$

The *Lie–Rinehart homology* is defined by

$$H^{\mathrm{Rin}}_n(L, \mathcal{M}) = H_n\big(\mathcal{M} \otimes_{\mathrm{A}} \Lambda^n_{\mathrm{A}}(L)\big), \qquad n \geq 0.$$

Cohomology

Definition

Let $\mathcal{M}$ a left (A, L)-module. Consider the cochain complex:

$$\partial\colon \operatorname{Hom}_{\mathrm{A}}(\Lambda^{n-1}_{\mathrm{A}} L, \mathcal{M}) \to \operatorname{Hom}_{\mathrm{A}}(\Lambda^{n}_{\mathrm{A}} L, \mathcal{M}),$$

$$(\partial f)(x_1, \ldots, x_n) = \sum_{i=1}^{n} (-1)^{(i-1)} x_i\big(f(x_1, \ldots, \hat{x}_i, \ldots, x_n)\big) + \sum_{j<k} (-1)^{j+k} f([x_j, x_k], x_1, \ldots, \hat{x}_j, \ldots, \hat{x}_k, \ldots, x_n),$$

The *Lie–Rinehart cohomology* is defined by

$$H^n_{\mathrm{Rin}}(L, \mathcal{M}) = H^n\big(\operatorname{Hom}_{\mathrm{A}}(\Lambda^n_{\mathrm{A}} L, \mathcal{M})\big), \qquad n \geq 0.$$

Low (Co)Homology Modules

- $H_0^{\mathrm{Rin}}(L, \mathcal{M}) = \dfrac{\mathcal{M}}{\mathcal{M} \circ L}$,
- $H^0_{\mathrm{Rin}}(L, \mathcal{M}) = \{m \in \mathcal{M} : xm = 0 \text{ for all } x \in L\}$,
- $H^1_{\mathrm{Rin}}(L, \mathcal{M}) = \frac{\mathrm{Der}_{\mathrm{A}}(L, \mathcal{M})}{\mathrm{IDer}_{\mathrm{A}}(L, \mathcal{M})}$.
- $H^2_{\mathrm{Rin}}(L, \mathcal{M})$ classifies the abelian extensions of L by $\mathcal{M}$. (Huebschmann, 1990)

Central Extensions

Definition

The *center* of a Lie–Rinehart algebra L is

$$Z_{\mathrm{A}}\, L = \{x \in L : [ax, z] = 0 \text{ and } x(a) = 0 \text{ for all } a \in \mathrm{A}, z \in L\}.$$

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Definition

Let L be a Lie–Rinehart algebra. A *central extension of L* is a surjective Lie–Rinehart homomorphism $p \colon E \longrightarrow\!\!\!\!\!\rightarrow L$ such that $\operatorname{Ker} p \subset Z_{\mathrm{A}} E$.

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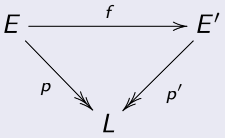
Definition

Let L be a Lie–Rinehart algebra. A *central extension of L* is a surjective Lie–Rinehart homomorphism $p\colon E \twoheadrightarrow L$ such that $\operatorname{Ker} p \subset \mathrm{Z}_{\mathrm{A}}\, E$.

Definition

A *homomorphism of central extensions* from $p\colon E \twoheadrightarrow L$ to $p\colon E' \twoheadrightarrow L$ is a commutative diagram of Lie–Rinehart homomorphisms



Classification

Proposition

If L is A-projective, then $H^2_{\mathrm{Rin}}(L, I)$ classifies the central extensions of L whose kernel is I.

$$0 \longrightarrow I \longrightarrow E \longrightarrow L$$

Universal Central Extension

Definition

A central extension $\mathfrak{u}\colon E \twoheadrightarrow L$ is a universal central extension if for any central extension $p'\colon E' \twoheadrightarrow L$ there exists a unique homomorphism of central extensions $f\colon E \to E'$

$$\begin{array}{ccc} E & \xrightarrow{\quad f \quad} & E' \\ & {\scriptstyle \mathfrak{u}} \searrow \quad \swarrow {\scriptstyle p'} & \\ & L & \end{array}$$

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Theorem

If $\mathfrak{u}\colon E \twoheadrightarrow L$ is a universal central extension, then E (and L) are perfect.

Functor $\mathfrak{uce}_{\mathrm{A}}$

Definition

Let $\mathfrak{uce}_{\mathrm{A}} L$ the A-module $\mathrm{A} \otimes L \otimes L$ subject to the relations

- $a \otimes x \otimes x$,
- $a \otimes x \otimes [y, z] + a \otimes y \otimes [z, x] + a \otimes z \otimes [x, y]$,
- $a \otimes [x, y] \otimes [x', y'] + [x, y](a) \otimes x' \otimes y' - 1 \otimes [x, y] \otimes a[x', y']$.

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Proposition

$\mathfrak{uce}_{\mathrm{A}}$ is a functor from $\mathcal{LR}_{\mathrm{AK}}$ to $\mathcal{LR}_{\mathrm{AK}}$, where for $f : L \to M$, $\mathfrak{uce}_{\mathrm{A}}(f) : \mathfrak{uce}_{\mathrm{A}} L \to \mathfrak{uce}_{\mathrm{A}} M$ is defined by $a \otimes x \otimes y \mapsto a \otimes f(x) \otimes f(y)$.

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Theorem (Castiglioni–G.M.–Ladra)

If L is perfect, then

$$0 \longrightarrow \ker \mathfrak{u} \longrightarrow \mathfrak{uce}_{\mathrm{A}} L \stackrel{\mathfrak{u}}{\longrightarrow} L$$

is the universal central extension of L.

Lie–Rinehart pairing

Definition

Let L and M be Lie–Rinehart algebras, an *action* of L on M is a K-linear map $L \times M \to M$, $(x, m) \mapsto {}^{x}m$, such that

- ${}^{x}(am) = a({}^{x}m) + x(a)m$,
- ${}^{[x,y]}m = {}^{x}({}^{y}m) - {}^{y}({}^{x}m)$,
- ${}^{x}[m, n] = [{}^{x}m, n] + [m, {}^{x}n]$.

Non-abelian Tensor Product

Definition

Let L and M two Lie–Rinehart algebras with an action of L on M and an action of M on L. The *non-abelian tensor product $L \otimes M$* is the A-module spanned by the symbols $x \otimes m$ subject to the relations:

- $k(x \otimes m) = kx \otimes m = x \otimes km$,
- $x \otimes (m + n) = x \otimes m + x \otimes n$,
  $(x + y) \otimes m = x \otimes m + y \otimes m$,
- $[x, y] \otimes m = x \otimes {}^{y}m - y \otimes {}^{x}m$,
  $x \otimes [m, n] = {}^{n}x \otimes m - {}^{m}x \otimes n$,
- $$\begin{aligned}[a(x \otimes m), b(y \otimes n)] = & \; a({}^{m}x) \otimes b({}^{y}n) \\ = & - ab({}^{m}x \otimes {}^{y}n) + a\alpha(x \otimes m)(b)(y \otimes n) \\ & - \alpha(y \otimes n)(a)b(x \otimes m).\end{aligned}$$

Non-abelian Tensor Product and Universal Central Extension

Theorem (Castiglioni–G.M.–Ladra)

If L is a perfect Lie–Rinehart algebra, then the non-abelian tensor product $L \otimes L$ where the action of L on L is the Lie bracket is the universal central extension of L.