Hardy spaces with variable exponents and generalized Campanato spaces

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Eiichi Nakai and Yoshihiro Sawano, Hardy spaces with variable exponents and generalized Campanato spaces, Journal of Functional Analysis **262** *(2012) 3665–3748*

and

Yoshihiro Sawano, Atomic decompositions of Hardy spaces with variable exponents and its application to bounded linear operators, Integral Equations Operator Theory **77** *(2013), no. 1, 123–148.*

Motivation and objective of this talk

1. Define the variable Hardy spaces as an extension of the variable Lebesgue spaces.
2. Obtain the atomic decomposition.
3. Generalized Campanato spaces.
4. Littlewood-Paley decomposition.
5. Duality.
6. Local Hardy spaces.
7. Some open problems.

My first aim is to study Hardy spaces with variable exponent of its own right. However, after studying Hardy spaces with variable exponent I noticed that I can mix the Hardy spaces with various parameters.
One of my aims today is to have unified understanding of complicated aspect of Hardy spaces.

1. The Hardy space H^p with $1 < p < \infty$ coincides with L^p.
2. The Hardy space H^1 is a proper subset of L^1.
3. The Hardy space H^p with $0 < p < 1$ is not contained in $L^1_{\rm loc}$.

Therefore, the theory of Hardy spaces is very complicated.

Hardy spaces and the related notations

Definition (The review of the classical definition)

1. Topologize $\mathcal{S}(\mathbb{R}^n)$ by the norms $\{p_N\}_{N\in\mathbb{N}}$ given by
$$p_N(\varphi) \equiv \sum_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} (1+|x|)^N |\partial^\alpha \varphi(x)|$$
for each $N \in \mathbb{N}$. Define $\mathcal{F}_N \equiv \{\varphi \in \mathcal{S}(\mathbb{R}^n) : p_N(\varphi) \le 1\}$.
2. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. The grand maximal operator $\mathcal{M}f$ is given by
$$\mathcal{M}f(x) \equiv \sup\{|t^{-n}\psi(t^{-1}\cdot)*f(x)| \,:\, t>0,\, \psi \in \mathcal{F}_N\} \, (x \in \mathbb{R}^n),$$
where we choose and fix a large integer N.
3. The Hardy space $H^p(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $\|f\|_{H^p} \equiv \|\mathcal{M}f\|_{L^p}$ is finite.

We want to mix L^{p_0} and L^{p_1} with $p_0 \neq p_1$. In particular let us consider $0 < p_0 < 1 < p_1 < \infty$. Then how do we achieve this ?

1. Use variable Lebesgue spaces. Vary the value of p according to the position of x.

$$\|f\|_{L^{p(\cdot)}} \equiv \inf \left\{ \lambda > 0 \, : \, \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

2. Use Orlicz spaces. Vary the value of p according to the value of $f(x)$.

$$\|f\|_{L^{\Phi}} \equiv \inf \left\{ \lambda > 0 \, : \, \int_{\mathbb{R}^n} \Phi \left(\frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}.$$

The function $p(\cdot)$ is called the variable exponent. It is customary to denote $p_+ \equiv \sup_{x \in \mathbb{R}^n} p(x)$ and $p_- \equiv \inf_{x \in \mathbb{R}^n} p(x)$, which we shall do throughout this talk. As is often the case with many other cases, we postulate on $p(\cdot)$ the following conditions.

(log-Hölder continuity)

$$|p(x) - p(y)| \lesssim \frac{1}{\log(1/|x-y|)} \quad \text{for} \quad |x-y| \le \frac{1}{2}, \tag{1}$$

(decay condition)

$$|p(x) - p(y)| \lesssim \frac{1}{\log(e+|x|)} \quad \text{for} \quad |y| \ge |x|. \tag{2}$$

Denote by p_∞ the limit $\lim_{x \to \infty} p(x)$ ensured by the decay condition.

Definition

The Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $\|f\|_{H^{p(\cdot)}} \equiv \|\mathcal{M}f\|_{L^{p(\cdot)}}$ is finite.

Definition ($(p(\cdot), q)$-atom)

Let $q \in [1, \infty]$. A function a is said to be a $(p(\cdot), q)$-atom if it is supported on a cube Q with the following properties.

1. (Size condition) $\|a\|_q \leq \dfrac{|Q|^{1/q}}{\|\chi_Q\|_{L^{p(\cdot)}}}$
2. (Moment condition) $\displaystyle\int_Q a(x)x^\alpha\,dx = 0$ for all

$$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$$

with $\|\alpha\|_1 \equiv \sum_{j=1}^n \alpha_j \leq n\left(\frac{1}{p_-} - 1\right)$.

Atomic decomposition of $L^{p(\cdot)}(\mathbb{R}^n)$

Theorem

Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then TFAE for any $q \in (p_+, \infty) \cap [1, \infty]$.

1. $f \in H^{p(\cdot)}(\mathbb{R}^n)$.
2. *There exists a sequence $\{a_j\}_{j=1}^{\infty}$ of $(p(\cdot), \infty)$-atoms such that $f = \sum_{j=1}^{\infty} \lambda_j a_j$ with*
$$\left\| \sum_{j=1}^{\infty} \frac{|\lambda_j| \chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}} \right\|_{L^{p(\cdot)}} < \infty.$$
3. *There exists a sequence $\{a_j\}_{j=1}^{\infty}$ of $(p(\cdot), q)$-atoms such that $f = \sum_{j=1}^{\infty} \lambda_j a_j$ with*
$$\left\| \left(\sum_{j=1}^{\infty} \frac{|\lambda_j|^{\min(1,p_-)} \chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}^{\min(1,p_-)}} \right)^{1/\min(1,p_-)} \right\|_{L^{p(\cdot)}} < \infty.$$

This theorem holds whenever $0 < p_- \leq p_+ < \infty$. This is what we said; we could unify the theory of Hardy spaces.

Molecular decomposition

Definition

Let $d_{p(\cdot)} \equiv \min \{d \in \mathbb{N} \cup \{0\} : p_-(n+d+1) > n\}$. Let $0 < p_- \leq p_+ < q \leq \infty$, $q \geq 1$ and $d \in [d_{p(\cdot)}, \infty) \cap \mathbb{Z}$ be fixed. One says that $\mathfrak{M}$ is a $(p(\cdot), q)$-molecule centered at a cube Q if it satisfies the following conditions.

1. On $2\sqrt{n}Q$, $\mathfrak{M}$ satisfies $\|\mathfrak{M}\|_{L^q(2\sqrt{n}Q)} \leq \dfrac{|Q|^{\frac{1}{q}}}{\|\chi_Q\|_{L^{p(\cdot)}}}$.
2. $|\mathfrak{M}(x)| \leq \dfrac{1}{\|\chi_Q\|_{L^{p(\cdot)}}} \left(1 + \dfrac{|x-z|}{\ell(Q)}\right)^{-2n-2d-3}$ outside $2\sqrt{n}Q$. This condition is called the decay condition.
3. If α is a multiindex with length less than d, then we have

$$\int_{\mathbb{R}^n} x^\alpha \mathfrak{M}(x)\, dx = 0.$$

Littlewood-Paley characterization

The molecular decomposition yields the Littlewood-Paley characterization in turn:

Theorem

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *be a function supported on* $Q(0,4) \setminus Q(0,1/4)$ *such that*

$$\sum_{j=-\infty}^{\infty} |\varphi_j(\xi)|^2 > 0$$

for $\xi \in \mathbb{R}^n \setminus \{0\}$*. Then the following norm is an equivalent norm of* $H^{p(\cdot)}(\mathbb{R}^n)$*:*

$$\|f\|_{\dot{F}^0_{p(\cdot)2}} \equiv \left\| \left(\sum_{j=-\infty}^{\infty} |\varphi_j(D)f|^2 \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot)}}, \quad f \in \mathcal{S}'(\mathbb{R}^n). \qquad (3)$$

Generalized Campanato spaces

$\mathcal{P}_d(\mathbb{R}^n)$ is the set of all polynomials having degree at most d. For a locally integrable function f, a cube Q and a nonnegative integer d, there exists a unique polynomial $P \in \mathcal{P}_d(\mathbb{R}^n)$ such that, for all $q \in \mathcal{P}_d(\mathbb{R}^n)$,

$$\int_Q (f(x) - P(x))q(x)\,dx = 0.$$

Denote this unique polynomial P by $P_Q^d f$.

Definition

Define $d_{p(\cdot)} \equiv \min\{d \in \mathbb{N} \cup \{0\} : p_-(n+d+1) > n\}$. Let $L^q_{\rm comp}(\mathbb{R}^n)$ be the set of all L^q-functions with compact support. For a nonnegative integer d, let

$$L^{q,d}_{\rm comp}(\mathbb{R}^n) \equiv \left\{ f \in L^q_{\rm comp}(\mathbb{R}^n) : \int_{\mathbb{R}^n} f(x) x^\alpha \, dx = 0, \ |\alpha| \le d \right\}.$$

Likewise if Q is a cube, then we write

$$L^{q,d}(Q) \equiv \left\{ f \in L^q(Q) : \int_Q f(x) x^\alpha \, dx = 0, \ |\alpha| \le d \right\}.$$

Campanato space

Definition ($\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$)

Let $\phi : \mathcal{Q} \to (0, \infty)$ be a function and $f \in L^q_{\rm loc}(\mathbb{R}^n)$. One denotes

$$\|f\|_{\mathcal{L}_{q,\phi,d}} = \sup_{Q \in \mathcal{Q}} \frac{1}{\phi(Q)} \left(\frac{1}{|Q|} \int_Q |f(x) - P_Q^d f(x)|^q \, dx \right)^{1/q},$$

when $q < \infty$ and

$$\|f\|_{\mathcal{L}_{q,\phi,d}} = \sup_{Q \in \mathcal{Q}} \frac{1}{\phi(Q)} \|f - P_Q^d f\|_{L^\infty(Q)}.$$

when $q = \infty$. Then the Campanato space $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ is defined to be the sets of all f such that $\|f\|_{\mathcal{L}_{q,\phi,d}} < \infty$.

Remark

Here and below we make a slight abuse of notation. We write

$$\phi(x,r) \equiv \phi(Q(x,r))$$

for $x \in \mathbb{R}^n$ and $r > 0$. For $Q \in \mathcal{Q}$ and $f \in L^q(Q)$, $\|P_Q^d f\|_{L^\infty(Q)} \lesssim \left(\frac{1}{|Q|}\int_Q |f(x)|^q \, dx\right)^{\frac{1}{q}}$, where the implicit constant in $\lesssim$ does not depend on $Q \in \mathcal{Q}$ and $f \in L^q(Q)$. Hence we see

$$\|f\|_{\mathcal{L}_{q,\phi,d}} \sim \sup_{Q \in \mathcal{Q}} \inf_{P \in \mathcal{P}_d(\mathbb{R}^n)} \frac{1}{\phi(Q)} \left(\frac{1}{|Q|}\int_Q |f(x) - P(x)|^q \, dx\right)^{1/q}.$$

Examples

Here is some examples of the function ϕ we envisage.

Example

Let u be a real number in $(0, \infty)$.

(1) $\phi_1(Q) = |Q|^{\frac{1}{u}-1}$. In this case $\mathcal{L}_{p,\phi_1,d}$ is known to be the Lipschitz space when $u < 1$ and the BMO space when $u = 1$.

(2) $\phi_2(Q) = \dfrac{|Q|^{\frac{1}{u}} + |Q|}{|Q|} = \phi_1(Q) + 1$.

(3) $\phi_3(Q) = \dfrac{\|\chi_Q\|_{L^{p(\cdot)}}}{|Q|}$.

We can consider the function space $\mathcal{L}_{q,\phi,d}(\mathbb{R}^n)$ in a wide generality. It often turns out that the following conditions suffice.

(A1) There exists a constant $C > 0$ such that

$$C^{-1} \le \frac{\phi(x,r)}{\phi(x,2r)} \le C, \quad (x \in \mathbb{R}^n,\ r > 0).$$

(A2) There exists a constant $C > 0$ such that

$$C^{-1} \le \frac{\phi(x,r)}{\phi(y,r)} \le C, \quad (x, y \in \mathbb{R}^n,\ r > 0,\ |x-y| \le r).$$

Example

If $p(\cdot)$ satisfies $0 < p_- \le p_+ < \infty$, (1) and (2), then ϕ_3 does satisfy (A1) and (A2).

With the help of the atomic decomposition, we can obtain duality.

Theorem

Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$, $0 < p_- \le p_+ \le 1$, $p_+ < q \le \infty$ *and* $1/q + 1/q' = 1$. *Suppose that the integer* d *is as in* (**??**). *Define*

$$\phi_3(Q) \equiv \frac{\|\chi_Q\|_{L^{p(\cdot)}}}{|Q|} \quad (Q \in \mathcal{Q}). \tag{4}$$

If $p(\cdot)$ *satisfies* (1) *and* (2), *then*

$$(H^{p(\cdot),q}_{\rm atom}(\mathbb{R}^n))^* \simeq \mathcal{L}_{q',\phi_3,d}(\mathbb{R}^n)$$

with equivalent norms. In particular, when q *is large enough, we have*

$$(H^{p(\cdot)}(\mathbb{R}^n))^* \simeq \mathcal{L}_{q',\phi_3,d}(\mathbb{R}^n).$$

Local Hardy spaces

Define the $h^{p(\cdot)}(\mathbb{R}^n)$ norm by:

$$\|f\|_{h^{p(\cdot)}} = \left\| \sup_{0<t<1} \sup_{\varphi \in \mathcal{F}_N} |t^{-n}\varphi(t^{-1}\cdot) * f| \right\|_{L^{p(\cdot)}}. \tag{5}$$

Then we have an analogy to $H^{p(\cdot)}(\mathbb{R}^n)$ and $h^{p(\cdot)}(\mathbb{R}^n)$ coincides with the Triebel-Lizorkin space $F^0_{p(\cdot),2}(\mathbb{R}^n)$ defined by Diening, Hasto and Roudenko.

Applications can be staged in many other spaces:

1. Morrey spaces. (Jointly with Tanaka and Iida)
2. Orlicz spaces. (Jointly with Nakai)
3. Their weighted variants. (In Orlicz spaces, this is jointly with Nakai).
4. Generalized quasi-Banach function spaces. (Jointly with Kwok-Pun-Ho).

Some open problems

Concerning the theory of variable exponents, I could not solve the following problems:

Dual spaces

I could not specify the dual space of $H^{p(\cdot)}$ when $0 < p_- \leq 1 \leq p_+ < \infty$.

Maximal operators

For a cube Q and an exponent $r(\cdot) : \mathbb{R}^n \to (1, \infty)$ such that $1 < r_- \leq r_+ < \infty$, define

$$m_Q^{r(\cdot)}(f) := \inf \left\{ \lambda > 0 \,:\, \frac{1}{|Q|} \int_Q \left(\frac{|f(x)|}{\lambda} \right)^{r(x)} dx \leq 1 \right\}$$

and

$$M_{r(\cdot)} f(x) := \sup_{Q \in \mathcal{Q}} \chi_Q(x) m_Q^{r(\cdot)}(f).$$

Assuming that $1 < (p(\cdot)/r(\cdot))_- \leq (p(\cdot)/r(\cdot))_+ < \infty$, can we prove

$$\|M^{r(\cdot)} f\|_{L^{p(\cdot)}} \leq C \|f\|_{L^{p(\cdot)}}?$$

Morrey spaces

For $0 < q < p < \infty$, the Morrey norm is given by:

$$\|f\|_{\mathcal{M}^p_q} := \sup_{Q \in \mathcal{Q}} |Q|^{\frac{1}{p} - \frac{1}{q}} \left(\int_Q |f(y)|^q \, dy \right)^{1/q}.$$

Let (X, d, μ) be a metric mesure space, or suppose $(X, d, \mu) = (\mathbb{R}^n, |\cdot|, dx)$. What is the suitable definition of Morrey spaces with variable exponents ?

1. It seems to me; when the set X is bounded, then every plausible definition turns out to be the same.
2. What happens when X is not bounded.

Morrey spaces

Why am I asking this ?
There are (at least) two different function spaces which are called “Orlicz-Morrey spaces".
For a cube Q, define (φ, Φ)-average over Q by:

$$\|f\|_{(\varphi,\Phi);Q} \equiv \inf \left\{ \lambda > 0 \,:\, \frac{1}{\varphi(|Q|)|Q|} \int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\}$$

and define its Φ-average over Q by:

$$\|f\|_{\Phi;Q} \equiv \inf \left\{ \lambda > 0 \,:\, \frac{1}{|Q|} \int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\}$$

Let $\varphi : (0, \infty) \to (0, \infty)$ and $\Phi : [0, \infty) \to [0, \infty)$ be functions. Define

$$\|f\|_{\mathcal{L}_{\varphi,\Phi}} \equiv \sup_{Q \in \mathcal{Q}} \|f\|_{(\varphi,\Phi);Q}.$$

The function space $\mathcal{L}_{\varphi,\Phi}(\mathbb{R}^n)$ is defined to be the Orlicz-Morrey space of the first kind as the set of all measurable functions f for which the norm $\|f\|_{\mathcal{L}_{\varphi,\Phi}}$ is finite.

Define

$$\|f\|_{\mathcal{M}_{\varphi,\Phi}} \equiv \sup_{Q\in\mathcal{Q}} \frac{1}{\varphi(|Q|)}\|f\|_{\Phi;Q}.$$

The function space $\mathcal{M}_{\varphi,\Phi}(\mathbb{R}^n)$ is defined to be the Orlicz-Morrey space of the second kind as the set of all measurable functions f for which the norm $\|f\|_{\mathcal{M}_{\varphi,\Phi}}$ is finite. Gala, Sawano and Tanaka proved that these two notions are different.

Thank you for your attention!