One-sided operators in grand variable exponent Lebesgue spaces

ALEXANDER MESKHI

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, Georgia

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Abstract

Our aim is to present the bounddeness results of one-sided maximal, singular and potential operators in grand variable exponent Lebesgue spaces (GVLS briefly). The same problem for commutators of one-sided Calderón–Zygmund and potential operators is also investigated. These spaces introduced in [Ko-Me-GMJ] (see also the forthcoming monograph [Ko-Me-Ra-Sa], Chapter 14) unify two non-standard function spaces: variable exponent Lebesgue space and grand Lebesgue space. In [Ko-Me-GMJ] the authors established the boundedness of maximal, Calderón–Zygmund and potential operators defined on quasimetric spaces with doubling measure in GVELS.

Preliminaries

Let $I = (a, b)$ be an interval and let p be a measurable function on I satisfying the condition

$$1 < p_- \leq p_+ < \infty, \tag{1}$$

where

$$p_- := \inf_I p; \quad p_+ := \sup_I p.$$

Let us denote by $P(I)$ the class of all exponents on I satisfying (1).

Grand variable exponent Lebesgue space

We denote by $L^{p(\cdot)}(I)$ the variable exponent Lebesgue space defined on I. Further, let $\theta > 0$. We denote by $L^{p(\cdot),\theta}(I)$ the grand variable exponent Lebesgue space on I. This is the class of all measurable functions $f : I \mapsto \mathbb{R}$ for which the norm

$$\|f\|_{L^{p(\cdot),\theta}(I)} := \sup_{0<\varepsilon<p_- -1} \varepsilon^{\frac{\theta}{p_- -\varepsilon}} \|f\|_{L^{p(x)-\varepsilon}(I)}$$

is finite.

Preliminaries

Together with the space $L^{p(\cdot),\theta}$ it is interesting to consider the space $\mathcal{L}^{p(\cdot),\theta}$ which is defined with respect to the norm

$$\|f\|_{\mathcal{L}^{p(\cdot),\theta}} := \sup_{0<\varepsilon<p_- -1} \left\|\varepsilon^{\frac{\theta}{p(x)-\varepsilon}} f\right\|_{L^{p(x)-\varepsilon}(I)}.$$

It is obvious that the following continuous embedding holds:

$$\mathcal{L}^{p(\cdot),\theta}(I) \hookrightarrow L^{p(\cdot),\theta}(I).$$

It is known (see [KoMe-GMJ]) that there is a function f such that $f \in L^{p(\cdot),\theta}(I)$ but $f \notin \mathcal{L}^{p(\cdot),\theta}(I)$.

Preliminaries

If $p = p_c$ is constant, then $L^{p(\cdot),\theta} = \mathcal{L}^{p(\cdot),\theta}$ and it is the grand Lebesgue space $L^{p_c),\theta}$ introduced in [Greco, Iwaniec and Sbordone]. In the case $p = p_c =$ const and $\theta = 1$, we have the Iwaniec–Sbordone space $L^{p_c)}$. The space $L^{p_c)}$ naturally arises, for example, when studying integrability problems of the Jacobian under minimal hypothesis (see Iwaniec and Sbordone]), while $L^{p_c),\theta}$ is related to the investigation of the nonhomogeneous n- harmonic equation div $A(x, \nabla u) = \mu$ (see [Greco, Iwaniec and Sbordone]).

Proposition. [Ko-Me-GMJ] (a) The spaces $L^{p(\cdot),\theta}(I)$ and $\mathcal{L}^{p(\cdot),\theta}(I)$ are complete.
(b) The closure of $L^{p(\cdot)}(I)$ in $L^{p(\cdot),\theta}(I)$ (resp. in $\mathcal{L}^{p(\cdot),\theta}(I)$) consists of those $f \in L^{p(\cdot),\theta}(I)$ (resp. $f \in \mathcal{L}^{p(\cdot),\theta}(I)$) for which $\lim\limits_{\varepsilon\to 0} \varepsilon^{\frac{\theta}{p_- - \varepsilon}} \|f(\cdot)\|_{L^{p(\cdot)-\varepsilon}(I)} = 0$ (resp. $\lim\limits_{\varepsilon\to 0} \|\varepsilon^{\frac{\theta}{p(\cdot)-\varepsilon}} f(\cdot)\|_{L^{p(\cdot)-\varepsilon}(I)} = 0$).

Grand variable exponent Lebesgue spaces

The following properties hold:
Let $p \in P(I)$. Then the following embeddings hold:

$$L^{p(\cdot)}(I) \hookrightarrow L^{p(\cdot),\theta}(I) \hookrightarrow L^{p(\cdot)-\varepsilon}(I), \;\; 0 < \varepsilon < p_- - 1;$$

$$L^{p(\cdot)}(I) \hookrightarrow \mathcal{L}^{p(\cdot),\theta}(I) \hookrightarrow L^{p(\cdot)-\varepsilon}(I), \;\; 0 < \varepsilon < p_- - 1.$$

Small variable exponent Lebesgue spaces

It can be checked that the associate space of $L^{p(\cdot),\theta}(I)$ denoted by $SL^{p(\cdot),\theta}(I)$ is the small variable exponent Lebesgue space which is a Banach function space consisting of those measurable $g : I \mapsto \mathbb{R}$ for which

$$\|g\|_{SL^{p(\cdot),\theta}(I)} = \sup_{0 \le \psi \le |g|} \|\psi\|_{L^{(p'(\cdot),\theta}(I)} < \infty,$$

where

$$\|\psi\|_{L^{(p'(\cdot),\theta}(I)} = \inf_{g=\sum_{k=1}^{\infty} g_k} \left\{ \inf_{0<\varepsilon<p_- -1} \varepsilon^{\frac{-\theta}{p_- -\varepsilon}} \|g_k\|_{L^{p(\cdot)-\varepsilon}(I)} \right\}.$$

We say that an exponent p belongs to the class $\mathcal{P}_-(I)$ if there exists a non-negative constant c_1 such that for a.e. $x \in I$ and a.e. $y \in I$ with $0 < x - y \leq 1/2$, the inequality

$$p(x) \leq p(y) + \frac{c_1}{\ln(1/(x-y))} \tag{2}$$

Holds. Further, we say that p belongs to $\mathcal{P}_+(I)$ if there exists a non-negative constant c_2 such that for a.e. $x \in I$ and a.e. $y \in I$ with $0 < y - x \leq 1/2$, the inequality

$$p(x) \leq p(y) + \frac{c_2}{\ln(1/(y-x))} \tag{3}$$

holds.

The class $\mathcal{P}_-(I)$ (resp. $\mathcal{P}_+(I)$) is strictly larger than the class of exponents satisfying the log-Hölder continuity condition: there is a positive constant A such that for all $x, y \in I$, $|x-y| < 1/2$,

$$|p(x) - p(y)| \leq \frac{A}{-\log|x-y|}. \tag{4}$$

If we denote the class of exponents satisfying the latter condition by $\mathcal{P}(I)$, then

$$\mathcal{P}(I) = \mathcal{P}_+(I) \cap \mathcal{P}_-(I).$$

It is easy to see that if p is a non-increasing function on I, then condition $p \in \mathcal{P}_+(I)$ is satisfied, while for non-decreasing p condition $p \in \mathcal{P}_-(I)$ holds.

Decay Condition at Infinity

We say that p satisfies the decay condition at infinity if there is a positive constant A_∞ such that

$$|p(x) - p(y)| \leq \frac{A_\infty}{\log(e + |x|)}$$

for all $x, y \in I$, $|y| > |x|$.

One-sided maximal operators

Let I be an open set in $\mathbb{R}$.

$$\big(\mathcal{M}f\big)(x) = \sup_{h>0} \frac{1}{2h} \int\limits_{I(x,h)} |f(t)|dt,$$

$$\big(\mathcal{M}_{-}f\big)(x) = \sup_{h>0} \frac{1}{h} \int\limits_{I_{-}(x,h)} |f(t)|dt,$$

$$\big(\mathcal{M}_{+}f\big)(x) = \sup_{h>0} \frac{1}{h} \int\limits_{I_{+}(x,h)} |f(t)|dt,$$

where $x \in I$ and

$$I_{+}(x,h) := [x, x+h] \cap I; \;\; I_{-}(x,h) := [x-h, x] \cap I; \;\; I(x,h) := [x-h, x+h] \cap I.$$

One-sided Maximal Operator

In [Edmunds, Kokilashvili and Meskhi, Math. Nachr, 2008] it was shown there exists a discontinuous function $p \in P(I)$ such that $\mathcal{M}_-$ (resp.$\mathcal{M}_+$) is bounded in $L^{p(\cdot)}(I)$ but $\mathcal{M}$ is not bounded in $L^{p(\cdot)}(I)$.

Boundedness of one-sided maximal operators

The boundedness of one-sided maximal, singular and potential operators in variable exponent Lebesgue spaces under the "the one-sided" local log-Hölder continuity condition and decay condition at infinity was established in [Edmunds, Kokilashvili and Meskhi, Math.Nachr, 2008]. For example, for the left maximal operator the following statement holds:

Theorem. *Let I be an interval in $\mathbb{R}$ and let $p \in P(I)$.*

(a) If I be a bounded interval and $p \in \mathcal{P}_-(I)$, then $\mathcal{M}_-$ is bounded in $L^{p(\cdot)}(I)$.

(b) If I be $\mathbb{R}$ or $\mathbb{R}_+$, and $p \in \mathcal{P}_-(I) \cap \mathcal{P}_\infty(I)$, then $\mathcal{M}_-$ is bounded in $L^{p(\cdot)}(\mathbb{R}_+)$.

One-sided maximal operators in $L^{p(\cdot),\theta}(I)$

Theorem. *Let* $I := (0, a)$, $0 < a < \infty$ *be a bounded interval and let* $\theta > 0$. *Suppose that* $p \in P(I)$.
(i) If $p \in \mathcal{P}_-(I)$, then the one-sided Hardy–Littlewood maximal operator $\mathcal{M}_-$ is bounded in $L^{p(\cdot),\theta}(I)$;
(ii) If $p \in \mathcal{P}_+(I)$, then the one-sided Hardy–Littlewood maximal operator $\mathcal{M}_+$ is bounded in $L^{p(\cdot),\theta}(I)$.

Regarding the space $\mathcal{L}^{p(\cdot),\theta}(I)$ we have the following statement.

Theorem. *Let I be a bounded interval and let $\theta > 0$.*

(i) If $p \in P(I) \cap \mathcal{P}_-(I)$, then the one-sided Hardy–Littlewood maximal operator $\mathcal{M}_-$ is bounded in $\mathcal{L}^{p(\cdot),\theta}(I)$;

(ii) If $p \in P(I) \cap \mathcal{P}_+(I)$, then the one-sided Hardy–Littlewood maximal operator $\mathcal{M}_+$ is bounded in $\mathcal{L}^{p(\cdot),\theta}(I)$.

(For the two-sided Hardy-Littlewood maximal operator see [Diening], [Cruz-Urive, Fiorenza, Neugebauer]; [Nekvinda]).

Calderón-Zygmund Kernel

Let $I := (-a, a)$, $0 < a \leq \infty$. We say that a function k in $L^1_{loc}(I \setminus \{0\})$ is a Calderón–Zygmund kernel if the following properties are satisfied:

(a) there exists a finite constant B_1 such that

$$\left| \int\limits_{\varepsilon<|x|<N} k(x)dx \right| \leq B_1$$

for all ε and all N, with $0 < \varepsilon < N < 2a$, and furthermore, $\lim\limits_{\varepsilon\to 0} \int\limits_{\varepsilon<|x|<N} k(x)dx$ exists;

(b) there exists a positive constant B_2 such that
$\big|k(x)\big| \leq \frac{B_2}{|x|}, \qquad x \in I \setminus \{0\};$

(c) there exists a positive constant B_3 such that for all $x, y \in I$ with $|x| > 2|y| > 0$ the inequality

$$|k(x-y) - k(x)| \leq B_3 \frac{|y|}{|x|^2}$$

Calderón-Zygmund Operators

It is known (see [Aimar, Forzani and Martin-Reyes]) that if $a = \infty$, (a)-(c) are satisfied for the kernel k defined on $\mathbb{R}$, then the operators

$$K^*f(x) = \sup_{\varepsilon>0} \left| K_\varepsilon f(x) \right|;$$

$$Kf(x) = \lim_{\varepsilon\to 0} K_\varepsilon f(x),$$

where

$$K_\varepsilon f(x) = \int\limits_{|x-y|>\varepsilon} k(x-y)f(y)dy,$$

have weak $(1,1)$ type and are bounded in $L^r(\mathbb{R})$, $1 < r < \infty$
It is clear that $Kf(x) \leq K^*f(x)$.

Calderón-Zygmund kernel

The following example shows the existence of a non-trivial Calderón-Zygmund kernel with a support contained in $(0, a)$ (see [Aimar, Firzani-Martin-Reyes] for $a = \infty$).

Example

The function

$$k(x) = \frac{1}{x} \, \frac{\sin(\ln x)}{\ln x} \chi_{(0,a)}(x)$$

is a Calderón–Zygmund kernel.

There exists also a non-trivial Calderón–Zygmund kernel supported in the interval $(-a, 0)$.

C-Z operators in $L^{p(\cdot),\theta}(I)$

Theorem. *Let* $I := (0, a)$ *be a bounded interval and let* $\theta > 0$. *Suppose that* $p \in P(I)$.
(i) If $p \in \mathcal{P}_+(I)$, then for the Calderón-Zygmund operator K with kernel supported on $(-2a, 0)$, there is a positive constant c such that for all bounded f defined on I the inequality

$$\|K^* f\|_{L^{p(\cdot),\theta}(I)} \leq c\|f\|_{L^{p(\cdot),\theta}(I)};$$

holds;
(ii) If $p \in \mathcal{P}_-(I)$, then the Calderón-Zygmund operator K with kernel supported on $(0, 2a)$, there is a positive constant c such that for all bounded f defined on I the inequality

$$\|K^* f\|_{L^{p(\cdot),\theta}(I)} \leq c\|f\|_{L^{p(\cdot),\theta}(I)};$$

holds.

One-sided potentials in GVELS

We also studied the boundedness of one-sided fractional integral operators $\mathcal{W}_\alpha$ and $\mathcal{R}_\alpha$ in grand variable exponent Lebesgue space $\widetilde{L}^{p(\cdot),\theta}(I)$ which is narrower than the space $L^{p(\cdot),\theta}(I)$.
To formulate the main result in this direction we introduce new classes of exponents related to the classes $\mathcal{P}_-(I)$ and $\mathcal{P}_+(I)$. The class $\widetilde{\mathcal{P}}_-^{\ell_-}(I)$ (resp. $\widetilde{\mathcal{P}}_+^{\ell_+}(I)$) is the class of all non-negative $p \in \mathcal{P}_-(I)$ (resp. $p \in \mathcal{P}_+(I)$) such that $0 \leq \ell_- := \sup c_1(p) < \infty$ (resp. $0 \leq \ell_+ := \sup c_2(p) < \infty$), where $c_1(p)$ (resp. $c_2(p)$) is the best possible constant in (2) (resp. in (3)).

One-sided potentials in GVELS

Let $p \in P(I)$ and let $\theta > 0$. We introduce new spaces $\widetilde{L}_{+}^{p(\cdot),\theta,\ell_+}(I)$ and $\widetilde{L}_{-}^{p(\cdot),\theta,\ell_-}(I)$ defined with respect to the norms

$$\|f\|_{\widetilde{L}_{+}^{p(\cdot),\theta,\ell_+}(I)} := \sup\left\{ \eta_+^{\frac{\theta}{p_- - \eta_+}} \|f\|_{L^{p(x)-\eta(x)}(I)} : \right.$$

$$\left. 0 < \eta_- \le \eta_+ < p_- - 1,\ p(\cdot) - \eta(\cdot) \in \mathcal{P}_+^{\ell_+}(I) \right\} < \infty;$$

$$\|f\|_{\widetilde{L}_{-}^{p(\cdot),\theta,\ell_-}(I)} := \sup\left\{ \eta_+^{\frac{\theta}{p_- - \eta_-}} \|f\|_{L^{p(x)-\eta(x)}(I)} : \right.$$

$$\left. 0 < \eta_- \le \eta_+ < p_- - 1,\ p(\cdot) - \eta(\cdot) \in \mathcal{P}_-^{\ell_-}(I) \right\} < \infty.$$

It can be checked that the spaces $\widetilde{L}_{+}^{p(\cdot),\theta,\ell_+}(I)$ and $\widetilde{L}_{-}^{p(\cdot),\theta,\ell_-}(I)$ are Banach spaces.

Potentials in GVELS

Let $\theta > 0$ and $p \in P(I)$. We denote by $\widetilde{\mathcal{P}}^{\ell}(I)$ the collection of those exponents $\eta \in \mathcal{P}(I)$ for which

$$0 \le \ell := \sup A(\eta) < \infty,$$

where $A(\eta)$ is the best possible constant in (4). The class denoted by $\widetilde{L}^{p(\cdot),\theta,\ell}(I)$ consists of measurable functions $f : I \to \mathbb{R}$ for which

$$\|f\|_{\widetilde{L}^{p(\cdot),\theta,\ell}(I)} := \sup \Big\{ \varepsilon_+^{\frac{\theta}{p_- - \varepsilon_+}} \|f\|_{L^{p(x)-\varepsilon(x)}(I)} :$$

$$0 < \varepsilon_- \le \varepsilon_+ < p_- - 1,\ p(\cdot) - \varepsilon(\cdot) \in \widetilde{\mathcal{P}}^{\ell}(I) \Big\} < \infty.$$

One-sided potentials in GVELS

Like the spaces $\widetilde{L}_{+}^{p(\cdot),\theta,\ell_+}(I)$ and $\widetilde{L}_{-}^{p(\cdot),\theta,\ell_-}(I)$, the space $\widetilde{L}^{p(\cdot),\theta,\ell}(I)$ is a Banach space.

If $p = const$ and $\ell_{\pm} = 0$, then the space $\widetilde{L}^{p(\cdot),\theta,\ell}(I)$ is constant exponent grand Lebesgue spaces.

Let $I = (a, b)$ be a bounded interval in $\mathbb{R}$. We define the following potential operators on I:

$$\mathcal{W}^{\alpha} f(x) = \int_{x}^{b} f(t)(t-x)^{\alpha-1}dt, \qquad x \in I,$$

$$\mathcal{R}^{\alpha} f(x) = \int_{a}^{x} f(t)(x-t)^{\alpha-1}dt, \qquad x \in I;$$

$$\mathcal{I}^{\alpha} f(x) = \int_{a}^{b} f(t)|x-t|^{\alpha-1}dt, \qquad x \in I.$$

Boundedness of one-sided potentials in VELS

Theorem. *Let $p \in P(I)$ and let $\theta > 0$. Suppose that α is a constant such that $0 < \alpha < 1/p_+$. We set $q(x) = \frac{p(x)}{1-\alpha p(x)}$. Then*

(i) The operator $\mathcal{W}^\alpha$ is bounded from $\widetilde{L}_+^{p(\cdot),\theta,\ell_+}(I)$ to $\widetilde{L}_+^{q(\cdot),\frac{\theta q_-}{p_-},\widetilde{\ell}_+}(I)$;

(ii) The operator $\mathcal{R}^\alpha$ is bounded from $\widetilde{L}_-^{p(\cdot),\theta,\ell_-}(I)$ to $\widetilde{L}_-^{q(\cdot),\frac{\theta q_-}{p_-},\widetilde{\ell}_-}(I)$.

(iii) The operator $\mathcal{I}^\alpha$ is bounded from $\widetilde{L}^{p(\cdot),\theta,\ell}(I)$ to $\widetilde{L}^{q(\cdot),\frac{\theta q_-}{p_-},\widetilde{\ell}}(I)$, where

$$\widetilde{\ell}_\pm = \frac{\ell_\pm}{(1-\alpha p_+)^2}; \;\; \widetilde{\ell} = \frac{\ell}{(1-\alpha p_+)^2}.$$

Remark. If $p = p_c = const$ and $\ell = 0$, then the second parameter in the target space $\theta q_c/p_c$ is sharp in the sense that we can not replace it by smaller parameter (see [Me] for details).

One-sided Maximal and Calderón–Zygmund operators in $\widetilde{\mathcal{L}}_{-}p(\cdot), \theta, \ell_{-}(I)$ and $\widetilde{\mathcal{L}}_{+}p(\cdot), \theta, \ell_{+}(I)$

Theorem. *Let $p \in P(I)$ and let $\theta > 0$. Then*
(i) The operator $\mathcal{M}_{+}$ is bounded in $\widetilde{L}_{+}^{p(\cdot),\theta,\ell_{+}}(I)$;
(ii) The operator $\mathcal{M}_{-}$ is bounded in $\widetilde{L}_{-}^{p(\cdot),\theta,\ell_{-}}(I)$;
(iii) For the Calderón-Zygmund operator K with kernel supported on $(-2a, 0)$, there is a positive constant c such that for all bounded f defined on I the inequality

$$\|K^{*}f\|_{\widetilde{L}_{+}^{p(\cdot),\theta,\ell_{+}}(I)} \leq c\|f\|_{\widetilde{L}_{+}^{p(\cdot),\theta,\ell_{+}}(I)};$$

holds;
(iv) For the Calderón-Zygmund operator K with kernel supported on $(0, 2a)$, there is a positive constant c such that for all bounded f defined on I the inequality

$$\|K^{*}f\|_{\widetilde{L}_{-}^{p(\cdot),\theta,\ell_{-}}(I)} \leq c\|f\|_{\widetilde{L}_{-}^{p(\cdot),\theta,\ell_{-}}(I)};$$

holds.

The techniques and methods used in the proofs of the main statements enable us to derive the boundedness of commutators of Calderón-Zygmund singular integrals on I:

$$(K_b^{+,k}f)(x) = p.v. \int_I (b(x)-b(y))^k k(x-y)f(y)dy; \; supp\, k \subset (-2a, 0);$$

$$(K_b^{-,k}f)(x) = p.v. \int_I (b(x)-b(y))^k k(x-y)f(y)dy; \; supp\, k \subset (0, 2a),$$

where $b \in BMO(I), k = 0, 1, 2, \cdots$.
See [Lorente-Riveros].

Commutators of Calderón-Zygmund Singular Integrals

Theorem. *Let $I := (0, a)$ be a bounded interval and let $\theta > 0$. Suppose that $p \in P(I)$ and $b \in BMO(I)$.*
(i) If $p \in \mathcal{P}_+(I)$ and k be the Calderón-Zygmund kernel supported on $(-2a, 0)$, then there is a positive constant c such that for all bounded f the inequality

$$\|K_b^{+,k} f\|_{L^{p(\cdot),\theta}(I)} \le c\|b\|_{BMO}^k \|f\|_{L^{p(\cdot),\theta}(I)};$$

holds;
(ii) If $p \in \mathcal{P}_-(I)$ and k be the Calderón-Zygmund kernel supported on $(0, 2a)$, then there is a positive constant c such that for all bounded f the inequality

$$\|K_b^{-,k} f\|_{L^{p(\cdot),\theta}(I)} \le c\|b\|_{BMO}^k \|f\|_{L^{p(\cdot),\theta}(I)};$$

holds.
(iii) If k is the Calderón-Zygmund kernel supported on $(-2a, 0)$, then there is a positive constant c such that for all bounded f the inequality

$$\|K_b^{+,k} f\|_{\widetilde{\mathcal{L}}_+^{p(\cdot),\theta,\ell_+}(I)} \le c\|b\|_{BMO}^k \|f\|_{\widetilde{\mathcal{L}}_+^{p(\cdot),\theta,\ell_+}(I)};$$

(iv) If k is the Calderón-Zygmund kernel supported on $(0, 2a)$, then there is a positive constant c such that for all bounded f the inequality

$$\|K_b^{-,k} f\|_{\widetilde{\mathcal{L}}_-^{p(\cdot),\theta,\ell_-}(I)} \leq c\|b\|_{BMO}^k \|f\|_{\widetilde{\mathcal{L}}_-^{p(\cdot),\theta,\ell_-}(I)}.$$

holds.

The boundedness of commutators of one-sided fractional integrals is also studied.