Variable exponent Lorentz spaces

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Outline

1. Lorentz spaces

- Usual Lorentz spaces and equivalent norms
- The spaces $\ell_{q(\cdot)}(L_{p(\cdot)})$
- The Definition of $L_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$

2. Real interpolation

- Positive result
- Variable Marcinkiewicz
- The maximal operator and variable exponent Lebesgue spaces

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Classical Lorentz spaces

Let $f : \mathbb{R}^n \to \mathbb{C}$ be a measurable function.

distribution function $\mu_f : [0,\infty) \to [0,\infty]$ is defined by

$$\mu_f(s) = |\{x \in \mathbb{R}^n : |f(x)| > s\}|$$

non-increasing rearrangement $f^* : [0,\infty) \to [0,\infty]$ is defined by

$$f^*(t) = \inf\{s > 0 : \mu_f(s) \le t\}$$

Then for $0 < p, q \le \infty$ the Lorentz spaces $L_{p,q}(\mathbb{R}^n)$ is the collection of all measurable functions f such that the norm

$$\|f|L_{p,q}(\mathbb{R}^n)\| = \begin{cases} \left(\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t}\right)^{1/q} & \text{if } q < \infty \\ \sup_{t>0} t^{1/p} f^*(t) & \text{if } q = \infty \end{cases}$$

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Properties

- $L_{p,q}(\mathbb{R}^n)$ are complete and quasi-normed, i.e. quasi Banach spaces
- For $1 < p \leq \infty$ and $1 \leq q \leq \infty$ there exists an equivalent norm $\rightsquigarrow$ Banach spaces
- $L_{p,p}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$
- $L_{p,q_1}(\mathbb{R}^n) \subset L_{p,q_2}(\mathbb{R}^n)$ for $q_1 \leq q_2$
- $L_{\infty,q}(\mathbb{R}^n) = \{0\}$ for $q < \infty$ and $L_{\infty,\infty}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$

A first try for variable Lorentz spaces

Ephremidze, Kokilashvili, Samko '06
Israfilov, Kokilashvili, Tuzkaya '08
Take

$$\| f | L_{p,q}(\mathbb{R}^n) \| = \left\| t^{\frac{1}{p} - \frac{1}{q}} f^*(t) \right| L_q \left((0,\infty), \frac{dt}{t} \right) \Big\|$$

and make it variable by

$$\| f | \mathcal{L}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \| = \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)}} f^*(t) \right| L_{q(\cdot)} \left((0,\infty), \frac{dt}{t} \right) \Big\| .$$

Good: For $p(\cdot) = p$ and $q(\cdot) = q$ constant functions we obtain $\mathcal{L}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = L_{p,q}(\mathbb{R}^n)$
Bad: $\mathcal{L}_{p(\cdot),p(\cdot)}(\mathbb{R}^n) = L_{p(\cdot)}(\mathbb{R}^n)$ can not hold, since $L_{p(\cdot)}(\mathbb{R}^n)$ is not translation invariant.
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Equivalent norm on $L_{p,q}(\mathbb{R}^n)$

$$\|f|\,L_{p,q}(\mathbb{R}^n)\| = \left(\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t}\right)^{1/q}$$
$$= p^{1/q}\left(\int_0^\infty \lambda^q \left\|\chi_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}}\right| L_p(\mathbb{R}^n)\right\|^q \frac{d\lambda}{\lambda}\right)^{1/q}$$
$$\sim p^{1/q}\left(\sum_{k=-\infty}^{\infty} \left\|2^k\chi_{\{x\in\mathbb{R}^n:|f(x)|>2^k\}}\right| L_p(\mathbb{R}^n)\right\|^q\right)^{1/q}$$

Then f belongs to $L_{p(\cdot),q}(\mathbb{R}^n)$ if

$$\left\|f\right|L_{p(\cdot),q}(\mathbb{R}^n)\right\| = \begin{cases} \left(\int_0^\infty \lambda^q \left\|\chi_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}}\right| L_{p(\cdot)}(\mathbb{R}^n)\right\|^q \frac{d\lambda}{\lambda}\right)^{1/q}, & q<\infty \\ \sup_{\lambda>0}\lambda\left\|\chi_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}}\right| L_{p(\cdot)}(\mathbb{R}^n)\right\| & q=\infty \end{cases}$$

is finite.

The spaces $\ell_{q(\cdot)}(L_{p(\cdot)})$ [Almeida/Hästö '10]

For a sequence of measurable functions (f_ν) we define the modular

$$\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu) = \sum_\nu \inf\left\{\lambda_\nu > 0 : \varrho_{p(\cdot)}\left(\frac{f_\nu}{\lambda_\nu^{1/q(\cdot)}}\right) \le 1\right\}.$$

If $q^+ < \infty$ or $q(\cdot) \le p(\cdot)$, we can replace that with the more intuitive expression

$$\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu) = \sum_\nu \left\| \varphi_{q(\cdot)}(|f_\nu|) \middle| L_{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n) \right\|.$$

The space $\ell_{q(\cdot)}(L_{p(\cdot)})$ consists of all sequences (f_ν) such that

$$\left\| f_\nu \middle| \ell_{q(\cdot)}(L_{p(\cdot)}) \right\| = \inf\{\mu > 0 : \varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu/\mu) \le 1\} \quad \text{is finite.}$$

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We had the equivalence

$$\|f|\,L_{p,q}(\mathbb{R}^n)\| = \left(\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t}\right)^{1/q}$$
$$\sim p^{1/q} \left(\sum_{k=-\infty}^{\infty} \left\| 2^k \chi_{\{x\in\mathbb{R}^n : |f(x)|>2^k\}} \right| L_p(\mathbb{R}^n) \Big\|^q \right)^{1/q} .$$

$\rightsquigarrow$ A measurable function f belongs to $L_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ if

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Properties:

- For every $0 < p^- \leq p^+ \leq \infty$ and $0 < q^- \leq q^+ \leq \infty$ the spaces $L_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ are complete and quasi-normed $\rightsquigarrow$ quasi Banach spaces.
- If $p(\cdot) = p < \infty$ and $q(\cdot) = q$, then $L_{p(\cdot),q(\cdot)} = L_{p,q}$ are the usual Lorentz spaces
- For every $q(\cdot)$ we have $L_{\infty,q(\cdot)} = L_\infty$ in contrast to $L_{\infty,q} = \{0\}$ for $0 < q < \infty$ for usual Lorentz spaces.
- If $0 < p^- \leq p(\cdot) \leq p^+ \leq \infty$, then $L_{p(\cdot),p(\cdot)} = L_{p(\cdot)}$
- If $q_0(\cdot) \leq q_1(\cdot)$, then $L_{p(\cdot),q_0(\cdot)} \hookrightarrow L_{p(\cdot),q_1(\cdot)}$

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Real interpolation

- X_0, X_1 quasi Banach spaces
- $0 < \theta < 1$ and $0 < q \leq \infty$

The **real interpolation space** $(X_0, X_1)_{\theta,q}$ fulfills

$$X_0 \cap X_1 \hookrightarrow (X_0, X_1)_{\theta,q} \hookrightarrow X_0 + X_1.$$

It is defined by the quasi-norm

$$\|x|(X_0,X_1)_{\theta,q}\| = \begin{cases} \left(\int_0^\infty t^{-\theta q} K(x,t)^q \frac{dt}{t}\right)^{1/q} \\ \sup_{t>0} t^{-\theta} K(x,t) \end{cases},$$

where $K(x,t) = \inf\{\|x_0|X_0\| + t\,\|x_1|X_1\| : x = x_0 + x_1\}$.
Well known: $(L_p, L_\infty)_{\theta,q} = L_{\tilde{p},q}$ with $\frac{1}{\tilde{p}} = \frac{1-\theta}{p}$

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A positive result

Theorem

Let $p(\cdot), q_0(\cdot)$ be variable exponents with $p^+ < \infty$. Further let $0 < q \leq \infty$ and $0 < \Theta < 1$ and put

$$\frac{1}{\tilde{p}(\cdot)} = \frac{1-\Theta}{p(\cdot)}, \text{ then}$$

$$\left(L_{p(\cdot),q_0(\cdot)}, L_\infty\right)_{\Theta,q} = L_{\tilde{p}(\cdot),q}.$$

Special case: Taking $q_0(\cdot) = p(\cdot)$ yields

$$\left(L_{p(\cdot)}, L_\infty\right)_{\Theta,q} = L_{\tilde{p}(\cdot),q}$$

$\rightsquigarrow$ The variable Lorentz spaces $L_{p(\cdot),q(\cdot)}$ naturally arise by real interpolation between $L_{p(\cdot)}$ and L_∞

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The maximal operator

The Hardy-Littlewood maximal operator $\mathcal{M}$ for $f \in L_1^{loc}(\mathbb{R}^n)$ is defined as

$$(\mathcal{M}f)(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| dy \, ,$$

where the supremum is taken over all Balls (Cubes) B which contain x. For $1 < p \leq \infty$ we have

$$\|\mathcal{M}f \,|\, L_p(\mathbb{R}^n)\| \leq c \, \|f \,|\, L_p(\mathbb{R}^n)\| \, .$$

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Sketch of the proof

- trivial: $\mathcal{M}$ is bounded from $L_\infty(\mathbb{R}^n)$ to $L_\infty(\mathbb{R}^n)$
- weak-$L_p(\mathbb{R}^n)$ is the set of all measurable functions such that

$$\| f | w - L_p(\mathbb{R}^n) \| = \sup_{\lambda > 0} \lambda |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|^{1/p} < \infty$$

- hard: $\mathcal{M}$ is bounded from $L_1(\mathbb{R}^n)$ to weak-$L_1(\mathbb{R}^n)$
- external result: Macinkiewicz interpolation

Notation: $L_{p,\infty}(\mathbb{R}^n) =$ weak-$L_p(\mathbb{R}^n)$ and it holds $L_p(\mathbb{R}^n) \subset L_{p,\infty}(\mathbb{R}^n)$

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Marcinkiewicz interpolation

Theorem

Let T be a sublinear operator which is bounded from L_{p_0} into $L_{q_0,\infty}$ and from L_{p_1} into $L_{q_1,\infty}$, where $0 < p_0 \neq p_1 \leq \infty$ and $0 < q_0 \neq q_1 \leq \infty$. Let $0 < \Theta < 1$ and put

$$\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}, \quad \frac{1}{q} = \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}.$$

If

$$p \leq q, \tag{1}$$

then T is also bounded from L_p into L_q.

Boundedness of $\mathcal{M}$ on $L_{p(\cdot)}(\mathbb{R}^n)$

Theorem (Cruz-Uribe, Diening, Fiorenza, Harjulehto, Hästö, Mizuta, Nekvinda, Neugebauer, Shimomura)

If $1/p \in C^{\log}(\mathbb{R}^n)$ *and* $1 < p^- \leq p^+ \leq \infty$, *then* $\mathcal{M} : L_{p(\cdot)}(\mathbb{R}^n) \to L_{p(\cdot)}(\mathbb{R}^n)$ *is bounded; ie.*

$$\left\| \mathcal{M}f \,|\, L_{p(\cdot)}(\mathbb{R}^n) \right\| \leq c \left\| f \,|\, L_{p(\cdot)}(\mathbb{R}^n) \right\| .$$

The proof of the above Theorem does not use interpolation. It is very technical and requires some variable version of Jensens inequality

$$\varphi_{p(y)}(\beta \mathcal{M}f(y)) \leq \mathcal{M}(\varphi_{p(\cdot)}(f))(y) + \mathcal{M}((e + |\cdot|)^{-m})(y)$$

for $0 < \beta < 1$ and large $m > 0$.

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Open question

This question for a Marcinkiewicz theorem in variable exponent Lebesgue spaces was posed in 2004 by Diening, Hästö and Nekvinda.
Question: Let T be a sublinear operator which is of weak type $(\pi_0(\cdot), \pi_0(\cdot))$ and of weak type $(\pi_1(\cdot), \pi_1(\cdot))$. Is T then bounded from $L_{\pi_\Theta(\cdot)}$ to $L_{\pi_\Theta(\cdot)}$ with

$$\frac{1}{\pi_\Theta(\cdot)} = \frac{1-\Theta}{\pi_0(\cdot)} + \frac{\Theta}{\pi_1(\cdot)}?$$

weak type $(\pi(\cdot), \pi(\cdot))$ means, there exist a constant such that

$$\lambda \left\| \chi_{\{x\in\mathbb{R}^n : |Tf(x)|>\lambda\}} \middle| L_{\pi(\cdot)} \right\| \le c \left\| f \middle| L_{\pi(\cdot)} \right\| ,$$

i.e. the operator T is bounded from $L_{\pi(\cdot)}$ into $L_{\pi(\cdot),\infty}$.

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weak type estimate for $L_{p(\cdot)}$

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Ingenious idea: We prove the validity of the Marcinkiewicz interpolation theorem on variable Lebesgue spaces. Furthermore, we gain together with the boundednesses

$$\mathcal{M} : L_{p(\cdot)}(\mathbb{R}^n) \to L_{p(\cdot),\infty}(\mathbb{R}^n) \quad \text{and}$$
$$\mathcal{M} : L_\infty(\mathbb{R}^n) \to L_\infty(\mathbb{R}^n)$$

the boundedness of $\mathcal{M}$ on $L_{p(\cdot)}(\mathbb{R}^n)$.

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the boundedness of $\mathcal{M}$ on $L_{p(\cdot)}(\mathbb{R}^n)$.

Negative Result

In general Marcinkiewicz Interpolation does not hold in the variable exponent setting, ie.

T . . . sublinear operator

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Then in general it does **not** hold:

$T : L_{\pi_\theta(\cdot)} \to L_{\pi_\theta(\cdot)}$ with $1/\pi_\theta(\cdot) = (1-\theta)/\pi_0(\cdot) + \theta/\pi_1(\cdot)$

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Marcinkiewicz interpolation

Theorem

Let T be a sublinear operator which is bounded from L_{p_0} into $L_{q_0,\infty}$ and from L_{p_1} into $L_{q_1,\infty}$, where $0 < p_0 \neq p_1 \leq \infty$ and $0 < q_0 \neq q_1 \leq \infty$. Let $0 < \Theta < 1$ and put

$$\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}, \quad \frac{1}{q} = \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}.$$

If $p \leq q$, then T is also bounded from L_p into L_q.

There exist a sublinear operator T and $0 < p_0 \neq p_1 \leq \infty$, $0 < q_0 \neq q_1 \leq \infty$ and $0 < \theta < 1$ such that

- $T : L_{p_0}([0,1]) \to L_{q_0,\infty}([0,1])$
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Counterexample

Use the counterexample T to usual Marcinkiewicz from above and define $\tilde{T}$ by

$$\tilde{T}f(x) := \begin{cases} T(\chi_{[0,1]}f)(x-1), & \text{if } x \in [1,2] \\ 0 & \text{if } x \in [0,1) \end{cases}.$$

Put

$$\pi_0(x) := \begin{cases} p_0, & x \in [0,1) \\ q_0, & x \in [1,2] \end{cases} \quad \text{and } \pi_1(x) := \begin{cases} p_1, & x \in [0,1) \\ q_1, & x \in [1,2] \end{cases}$$

then

- $\tilde{T}$ is weak type $(\pi_0(\cdot), \pi_0(\cdot))$
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Open question

The counterexample provided works for general sublinear operators.
If we add more structure to the operator, then Marcinkiewicz could still hold in the variable exponent setting.
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