The Riesz potential in generalized Orlicz spaces

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Porto 10.6.2015

The talk is based on my joint work with Peter Hästö.

Let us write

$$\mathcal{I}_\alpha f(x) := \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy, \quad 0 < \alpha < n.$$

We want to show that $\|\mathcal{I}_\alpha f\|_{L^{\varphi_\alpha^\#(\cdot)}} \lesssim \|f\|_{L^{\varphi(\cdot)}}$.
We use Hedberg's method:

$$\begin{aligned}\mathcal{I}_\alpha f(x) &= \int_B \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy + \int_{\mathbb{R}^n \setminus B} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy \\ &\lesssim |B|^{\frac{\alpha}{n}} \mathcal{M} f(x) + |B|^{\frac{\alpha-n}{n}} \|\chi_B\|_{L^{\varphi^*(\cdot)}}.\end{aligned}$$

Let $\varphi \in N(\mathbb{R}^n)$ satisfy the following conditions

(A0) There exists $\beta \in (0,1)$ such that $1 \leqslant \varphi(x, \frac{1}{\beta})$ and $\varphi(x,\beta) \leqslant 1$,

(A1) There exists $\beta \in (0,1)$ such that $\beta\varphi^{-1}(x,t) \leqslant \varphi^{-1}(y,t)$ for every $t \in \left[1, \frac{1}{|B|}\right]$, every $x, y \in B$ and every ball B with $|B| \leqslant 1$.

(A2) $L^{\varphi(\cdot)}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) = L^{\varphi_\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, with $\varphi_\infty(t) := \limsup_{|x|\to\infty} \varphi(x,t)$.

(INC) There exists $\gamma > 1$ such that $s \mapsto s^{-\gamma}\varphi(x,s)$ is increasing for every $x \in \mathbb{R}^n$.

(DEC) There exists $\gamma < n$ such that $s \mapsto s^{-\frac{\gamma}{\alpha}}\varphi(x,s)$ is decreasing for every $x \in \mathbb{R}^n$.

We choose $t(x)$ such that $\varphi(x, t(x)) = 1$ and define

$$\varphi_1(x, s) := \varphi(x, t(x)s).$$

Now we have $\varphi_1(x, 1) \equiv 1$ and $\varphi_1 \simeq \varphi$ (equivalent) i.e. there exists $m > 1$ such that $\varphi_1(x, t/m) \leqslant \varphi(x, t) \leqslant \varphi_1(x, mt)$.
Next step

$$\varphi_2(x, t) := \max\{\varphi_1(x, t), 2t - 1\}$$

guarantees that $(\varphi_2)_\infty(t) := \limsup_{|x|\to\infty} \varphi_2(x, t)$ acts well when $t \leqslant 1$. We have $\varphi_2 \simeq \varphi_1 \simeq \varphi$.
Finally we set

$$\bar{\varphi}(x, t) = \begin{cases} 2\varphi_2(x, t) - 1, & \text{if} \quad t \geqslant 1, \\ (\varphi_2)_\infty(t), & \text{if} \quad t < 1. \end{cases}$$

Lemma.

$L^{\bar\varphi(\cdot)} = L^{\varphi(\cdot)}$ and the norms are comparable.

$\bar\varphi$ (and $\bar\varphi^*$) has the following good properties:

- $\bar\varphi \in N(\mathbb{R}^n)$
- $\bar\varphi(x,1) = 1$;
- $\bar\varphi(x,t) = \limsup_{|x|\to\infty} \bar\varphi(x,t)$ for $t \in [0,1]$;
- there exists $\beta \in (0,1)$ such that

$$\beta \bar\varphi^{-1}(x,t) \leqslant \bar\varphi^{-1}(y,t)$$

  for every $t \in \left[0, \frac{1}{|B|}\right]$, every $x, y \in B$ and every ball B.
- $\mathcal{M} : L^{\bar\varphi(\cdot)}(\mathbb{R}^n) \to L^{\bar\varphi(\cdot)}(\mathbb{R}^n)$ is bounded.

Proposition: point-wise estimate.

Then $\mathcal{I}_\alpha f(x) \lesssim \bar\varphi(x, \mathcal{M}f(x))^{-\frac{\alpha}{n}} \mathcal{M}f(x)$ a.e. for every $f \in L^{\varphi(\cdot)}(\mathbb{R}^n)$ with $\|f\|_{L^{\varphi(\cdot)}} \leqslant 1$.

Proof.

$$\begin{aligned} \mathcal{I}_\alpha f(x) &\lesssim |B|^{\frac{\alpha}{n}} \mathcal{M}f(x) + |B|^{\frac{\alpha-n}{n}} \|\chi_B\|_{\bar\varphi^*(\cdot)} \\ &\lesssim |B|^{\frac{\alpha}{n}} \mathcal{M}f(x) + \frac{|B|^{\frac{\alpha-n}{n}}}{\beta(\bar\varphi^*)^{-1}(x, \frac{1}{|B|})}. \end{aligned}$$

Now $(\bar\varphi^*)^{-1}(t) \approx t/\bar\varphi^{-1}(t)$ and so

$$\mathcal{I}_\alpha f(x) \lesssim |B|^{\frac{\alpha}{n}} \mathcal{M}f(x) + |B|^{\frac{\alpha}{n}} \bar\varphi^{-1}(x, \tfrac{1}{|B|}).$$

The claim follows by choosing $|B| = 1/\bar\varphi(x, \mathcal{M}f(x))$. □

Lemma.

Let $\lambda(x,t) := t\varphi(x,t)^{-\frac{\alpha}{n}}$. Then $\varphi \circ (\lambda^{-1})(x,\cdot)$ is equivalent to a convex Φ-function for every $x \in \mathbb{R}^n$.

Definition: the target space.

We define $\lambda(x,t) := t\varphi(x,t)^{-\frac{\alpha}{n}}$ and let $\varphi_\alpha^\# \in \Phi(\mathbb{R}^n)$ be the generalized Φ-function equivalent to $\varphi \circ (\lambda^{-1})$ given by the previous lemma.

Lemma.

Then $L^{\varphi_\alpha^\#(\cdot)}(\mathbb{R}^n) = L^{\bar{\varphi}_\alpha^\#(\cdot)}(\mathbb{R}^n)$ and the norms are comparable.

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Theorem: norm estimate.

Let $\varphi \in N(\mathbb{R}^n)$ satisfy assumptions (A0)–(A2), (INC), and (DEC) . Then

$$\|\mathcal{I}_\alpha f\|_{L^{\varphi^\#_\alpha(\cdot)}} \lesssim \|f\|_{L^{\varphi(\cdot)}}.$$

Proof. The point-wise estimate gives

$$\bar{\lambda}^{-1}(x, \mathcal{I}_\alpha f(x)) \lesssim \mathcal{M}(f(x)).$$

The claim follows by taking $\bar{\varphi}$ from the bouth sides and using the boundedness of the maximal operator. □

The norm estimate gives Sobolev-type inequalities.
Since $|u| \lesssim I_1|\nabla u|$ for $u \in W_0^{1,1}(\mathbb{R}^n)$, we obtain:

$$\|u\|_{L^{\varphi_1^{\#}(\cdot)}} \lesssim \|\nabla u\|_{L^{\varphi(\cdot)}} \quad \text{for all} \quad u \in W_0^{1,\varphi(\cdot)}(\mathbb{R}^n).$$

If $\Omega \subset \mathbb{R}^n$ is a John domain, then $|u - u_\Omega| \lesssim I_1|\nabla u|$, and thus

$$\|u - u_\Omega\|_{L^{\varphi_1^{\#}(\cdot)}(\Omega)} \lesssim \|\nabla u\|_{L^{\varphi(\cdot)}(\Omega)} \quad \text{for all} \quad u \in W^{1,\varphi(\cdot)}(\Omega).$$

The result is the best possible in the following sense: Let $\psi : [0,\infty) \to [0,\infty)$ be such that

$$\lim_{t\to\infty} \frac{\psi^{-1}(t)}{\lambda^{-1}(t)} = \infty.$$

Then there does not exists a constant $c > 0$ such that

$$\|u\|_{L^{\varphi\circ\psi^{-1}}(B(0,1))} \leqslant c \|\nabla u\|_{L^{\varphi}(B(0,1))}$$

for all $u \in W_0^{1,\varphi}(B(0,1))$.

Our submitted manuscript

P. Harjulehto and P. Hästö: The Riesz potential in generalized Orlicz spaces, March 2015,

can be download from

http://www.mathstat.helsinki.fi/analysis/varsobgroup/

Thank you!