Osnabrück, Germany • Lars Diening

# Numerical analysis for variable exponent problems

based on joint work with

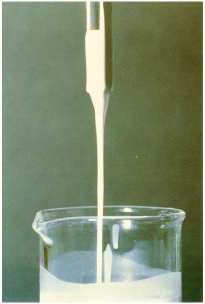
Berselli, Breit, Schwarzacher

# Electrorheological fluids

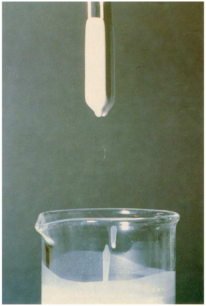
## Discovered by Winslow '49

Electrical field changes viscosity significantly! [play movie]

(1000 V/mm $\Leftrightarrow$ viscosity$\times$1000)



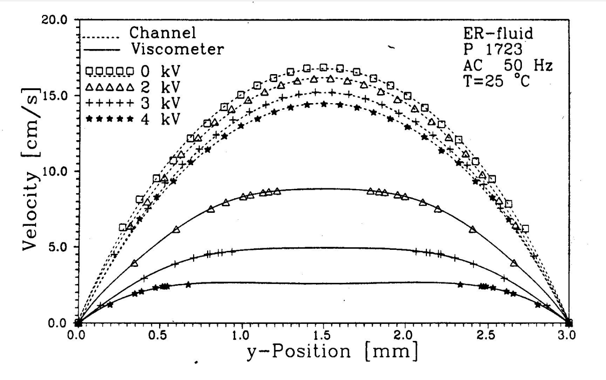
electrical field is OFF



electrical field is ON

# Electrorheological fluids – plug flow

## Experiment by Wunder




Conclusion: Extra stress (friction) depends on electrical field.

# Electrorheological fluids – model

## Model by Rajagopal and Růžička '96:

$$\begin{aligned}\partial_t \mathbf{v} - \operatorname{div} \mathbf{S} + \nabla q + [\nabla \mathbf{v}]\mathbf{v} &= \mathbf{f},\\ \operatorname{div} \mathbf{v} &= 0,\end{aligned}$$

with velocity $\mathbf{v}$, pressure $q$, extra stress $\mathbf{S}$, electric field $\mathbf{E}$.

The equation for $\mathbf{E}$ decouples.

Moreover, $\mathbf{S} = \mathbf{S}\big(\mathbf{E}, \varepsilon(\mathbf{v})\big) = (1 + |\varepsilon(\mathbf{v})|)^{p(t,x)-2}\varepsilon(\mathbf{v})$.

with symmetric gradient $\varepsilon(\mathbf{v}) = \frac{\nabla \mathbf{v} + (\nabla \mathbf{v})^T}{2}$ and $p(t,x) = p(|\mathbf{E}(t,x)|)$.

## Goal

Numerical analysis for finite element solutions.

**Simplified problem:** (constant $p$, no convection, no pressure, gradients)

$$-\mathrm{div}\big(|\nabla \mathbf{v}|^{p-2}\nabla \mathbf{v}\big) = \mathbf{f} \qquad (p\text{-Laplacian})$$
$$\mathbf{v} = 0 \qquad \text{on } \partial\Omega$$

Let $X := W_0^{1,p}(\Omega)$ and $\mathbf{S}(\mathbf{Q}) = |\mathbf{Q}|^{p-2}\mathbf{Q}$.

## Weak solutions

Find $\mathbf{v} \in W_0^{1,p}(\Omega)$ with $\langle \mathbf{S}(\nabla \mathbf{v}), \nabla \boldsymbol{\xi}\rangle = \langle \mathbf{f}, \boldsymbol{\xi}\rangle$ for all $\boldsymbol{\xi} \in W_0^{1,p}(\Omega)$.

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- Conformal triangulation $\mathcal{T}_h$.
- Let $X_h := \{\mathbf{w}_h \in W_0^{1,1}(\Omega) \,:\, \text{piecewise linear}\} \subset W_0^{1,\infty}(\Omega) \subset X$.

## Discrete solutions

Find $\mathbf{v}_h \in X_h$ with $\langle \mathbf{S}(\nabla \mathbf{v}_h), \nabla \boldsymbol{\xi}_h\rangle = \langle \mathbf{f}, \boldsymbol{\xi}_h\rangle$ for all $\boldsymbol{\xi}_h \in X_h$.

# Natural quantities

How should be measure the error between $\mathbf{v}$ and $\mathbf{v}_h$?

- $\|\nabla \mathbf{v}_h - \nabla \mathbf{v}\|_p$ is not a good! (sup-optimal)
- Recall: $\mathbf{S}(\mathbf{A}) = |\mathbf{A}|^{p-2}\mathbf{A}$. Define $\mathbf{F}(\mathbf{A}) := |\mathbf{A}|^{\frac{p-2}{2}}\mathbf{A}$. Then

$$(\mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{B})) \cdot (\mathbf{A} - \mathbf{B}) \eqsim |\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})|^2.$$

## Natural distance

$$\langle \mathbf{S}(\nabla \mathbf{v}_h) - \mathbf{S}(\nabla \mathbf{v}), \nabla \mathbf{v}_h - \nabla \mathbf{v} \rangle \eqsim \|\mathbf{F}(\nabla \mathbf{v}_h) - \mathbf{F}(\nabla \mathbf{v})\|_2^2.$$

## Natural regularity

Difference quotient technique gives $\mathbf{F}(\nabla \mathbf{v}) \in W^{1,2}_{\mathrm{loc}}$: (not $\mathbf{v} \in W^{2,p}$!!!)

$$h^{-2}\langle \tau_h \mathbf{S}(\nabla \mathbf{v}), \tau_h \nabla \mathbf{v} \rangle \eqsim h^{-2}\|\tau_h \mathbf{F}(\nabla \mathbf{v})\|_2^2 \xrightarrow{h \to 0} \|\nabla \mathbf{F}(\nabla \mathbf{v})\|_2^2.$$

# A priori estimates

**Theorem (Liu, Barrett '94; Diening, Růžička '07)**

$$\|\mathbf{F}(\nabla \mathbf{v}) - \mathbf{F}(\nabla \mathbf{v}_h)\|_2 \lesssim \inf_{\mathbf{w}_h \in X_h} \|\mathbf{F}(\nabla \mathbf{v}) - \mathbf{F}(\nabla \mathbf{w}_h)\|_2.$$

Let $\varphi_a(t) \eqsim (a+t)^{p-2} t^2$, then

$$\begin{aligned} |\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})|^2 &\eqsim \varphi_{|\mathbf{A}|}(|\mathbf{A} - \mathbf{B}|), \\ |\mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{B})|^2 &\eqsim \varphi'_{|\mathbf{A}|}(|\mathbf{A} - \mathbf{B}|). \end{aligned}$$

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**Theorem (Ebmeyer, Liu '05; Diening, Růžička '07)**

$$\|\mathbf{F}(\nabla \mathbf{v}) - \mathbf{F}(\nabla \mathbf{v}_h)\|_2 \leq \|\mathbf{F}(\nabla \mathbf{v}) - \mathbf{F}(\nabla \Pi_h \mathbf{v})\|_2 \lesssim h \left\|\nabla \mathbf{F}(\nabla \mathbf{v})\right\|_2,$$

*where $\Pi_h : X \to X_h$ is Scott-Zhang interpolation (defined by local means).*

Use $|\mathbf{F}(\mathbf{A}) - \mathbf{F}(\mathbf{B})|^2 \eqsim \varphi_{|\mathbf{A}|}(|\mathbf{A} - \mathbf{B}|)$ and stability of $\Pi_h$ in Orlicz spaces.

Requires Jensen's inequality $\varphi\Big(⨍_Q |f|\,dx\Big) \leq ⨍_Q \varphi(|f|)\,dx.$

Consider the case of variable exponents.

$$
\begin{aligned}
-\operatorname{div}\big(|\nabla \mathbf{v}|^{p(\cdot)-2}\nabla \mathbf{v}\big) &= \mathbf{f} && \text{($p$-Laplacian)}\\
\mathbf{v} &= 0 \qquad \text{on } \partial\Omega
\end{aligned}
$$

Let $X := W_0^{1,p(\cdot)}(\Omega)$ and $\mathbf{S}(\mathbf{Q}) = |\mathbf{Q}|^{p(\cdot)-2}\mathbf{Q}$.

## Weak solutions

Find $\mathbf{v} \in W_0^{1,p(\cdot)}(\Omega)$ with $\langle \mathbf{S}(\nabla \mathbf{v}), \nabla \boldsymbol{\xi}\rangle = \langle \mathbf{f}, \boldsymbol{\xi}\rangle$ for all $\boldsymbol{\xi} \in W_0^{1,p(\cdot)}(\Omega)$.

# Finite elements for the $p(\cdot)$-Laplacian

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# log-Hölder continuity

We say that $\frac{1}{p} \in C^{\log}(\mathbb{R}^n)$ if and only if

Local: $$\left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \leq \frac{c_{\log}(p)}{\log\left(e + \frac{1}{|x-y|}\right)}$$ [Zhikov '87, Diening '02]

Global: $$\left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right| \leq \frac{c_{\log}(p)}{\log(e + |x|)}$$ [Cruz-Uribe+F+N '03],

Less than the "usual" Hölder continuity.

**Theorem (Diening '02, Cruz-Uribe+FN '03, DHHMS '09)**

$\frac{1}{p} \in C^{\log}$ *and* $\inf p > 1$, *then* $\|Mf\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}$.

# Consequence of log-Hölder

Let $\frac{1}{p} \in C^{\log}$. Then for all balls $B$ with $|B| \leq 1$

$$\|\chi_B\|_{p(\cdot)} \eqsim |B|^{\frac{1}{p(x)}} \qquad \text{for all } x \in B.$$

**Key estimate: [Diening '02, DHHMS '09, DHHR '11, DSch '13]**

Let $\frac{1}{p} \in C^{\log}$. Then for all $m > 0$ and all balls $B$ with $|B| \leq 1$,

$$\left(\fint_B |f|\,dy\right)^{p(x)} \leq c_m \fint_B |f|^{p(\cdot)}\,dy + c\, r_B^m$$

for all $f$ with $\fint_B |f|\,dy \lesssim r_B^{-m}$ and all $x \in B$.

This is a substitute for Jensen's inequality!

# Projection error

## Key estimate: [Diening '02, DHHMS '09, DHHR '11, DSch '13]

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for all $f$ with $\fint_B |f| \lesssim r_B^{-m}$ and all $x \in B$.

## Projection error – intermediate

Let $\frac{1}{p} \in C^{\log}$ then for $m > 0$

$$\big\|\mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \nabla \Pi_h \mathbf{v})\big\|_2^2 \lesssim c_m \sum_T \int_{S_T} \big|\mathbf{F}(\nabla \mathbf{v}) - \mathbf{F}(\cdot, \langle \nabla \mathbf{v}\rangle_{S_T})\big|^2\,dx + h^m.$$

Where $T$ is a triangle, $h := \operatorname{diam}(T)$ and $S_T := \{T \text{ plus neighbors}\}$.

# Velocity error

## Projection error – intermediate

Let $\frac{1}{p} \in C^{\log}$ then for $m > 0$

$$\left\| \mathbf{F}(\cdot \nabla \mathbf{v}) - \mathbf{F}(\cdot, \nabla \Pi_h \mathbf{v}) \right\|_2^2 \lesssim c_m \sum_T \int_{S_T} \left| \mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \langle \nabla \mathbf{v} \rangle_{S_T}) \right|^2 dx + h^m.$$

## Projection error – final

Let $p \in C^{0,\alpha}$ then

$$\left\| \mathbf{F}(\cdot, \nabla \mathbf{v}) - \mathbf{F}(\cdot, \nabla \Pi_h \mathbf{v}) \right\|_2 \lesssim h \left\| \nabla \mathbf{F}(\nabla \mathbf{v}) \right\|_2 + h^\alpha.$$

- $h^\alpha$ is due to $\mathbf{F}(\cdot, \langle \nabla \mathbf{v} \rangle_{S_T})$ vs. $\langle \mathbf{F}(\cdot, \nabla \mathbf{v}) \rangle_{S_T}$.
- It is possible to freeze $p(\cdot)$ on each triangle.

# Summary

Combining the above techniques we get for $\frac{1}{p} \in C^{0,\alpha}$.

**A priori estimates $p(\cdot)$-Laplacian [Breit, Diening, Schwarzacher '15]**

$$\|\mathbf{F}(\nabla \mathbf{v}_h) - \mathbf{F}(\nabla \mathbf{v})\|_2 \lesssim h^\beta \qquad \text{with } \beta = \min\{1, \alpha\},$$

- Possible to treat the fluid system (i.e. with $\varepsilon(\mathbf{v}_h)$ and pressure).

  [Joint work with Breit and Berselli.]

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## Outlook – Open problems

- Convection + instationary.
- Higher order elements (already problem for $p$-Laplacian: $C^{1,\alpha}$). Useful for inf-sup-condition.
- Error estimates in terms of $\mathbf{S}(\nabla \mathbf{v})$ (relates better to pressure).