Atomic Decompositions in Variable 2-Microlocal Spaces and Applications

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2015 AMS–EMS–SPM INTERNATIONAL MEETING

Porto, 11 June 2015

Based on joint work with A. Caetano

OUTLINE

- Variable integrability
- **2-microlocal spaces with variable integrability**
- **Atomic / molecular decompositions**
- Sobolev type embeddings
- Convergence / Applications

Variable exponents

$\mathcal{P}(\mathbb{R}^n)$ – the set of all measurable functions $p : \mathbb{R}^n \to (0, \infty]$ (essentially) bounded away from zero.

Notation: $p^+ = \operatorname{ess\,sup}_{\mathbb{R}^n} p(x)$ and $p^- = \operatorname{ess\,inf}_{\mathbb{R}^n} p(x)$.

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log-Hölder continuity of $g : \mathbb{R}^n \to \mathbb{R}$: there exists $c_{\log} > 0$ s.t.

$$|g(x) - g(y)| \leq \frac{c_{\log}}{\log(e + 1/|x-y|)}, \qquad \forall x, y \in \mathbb{R}^n. \tag{1.1}$$

log-decay of $g : \mathbb{R}^n \to \mathbb{R}$: there exist $g_\infty \in \mathbb{R}$ and $C_{\log} > 0$ s.t.

$$|g(x) - g_\infty| \leq \frac{C_{\log}}{\log(e + |x|)}, \qquad \forall x \in \mathbb{R}^n. \tag{1.2}$$

$\mathcal{P}^{\log}(\mathbb{R}^n)$ – the class of those exponents $p \in \mathcal{P}(\mathbb{R}^n)$ s.t. $\frac{1}{p}$ satisfies both conditions (1.1) and (1.2).

The variable Lebesgue space

For $p \in \mathcal{P}(\mathbb{R}^n)$, the space $L_{p(\cdot)}$ consists of all (complex or extended real-valued) measurable functions f on $\mathbb{R}^n$ such that

$$\varrho_{p(\cdot)}(f/\lambda) := \int_{\mathbb{R}^n} \phi_{p(x)}\left(\frac{|f(x)|}{\lambda}\right) dx < \infty$$

for some $\lambda > 0$, where

$$\phi_p(t) := \begin{cases} t^p & \text{if } p \in (0,\infty), \\ 0 & \text{if } p = \infty \text{ and } t \in [0,1], \\ \infty & \text{if } p = \infty \text{ and } t \in (1,\infty]. \end{cases}$$

$L_{p(\cdot)}$ becomes a quasi-Banach space with respect to the quasi-norm

$$\|f|L_{p(\cdot)}\| := \inf\left\{\lambda > 0 : \varrho_{p(\cdot)}(f/\lambda) \leq 1\right\}.$$

Variable exponent mixed sequence spaces

For $p, q \in \mathcal{P}(\mathbb{R}^n)$, the **mixed Lebesgue-sequence space** $L_{p(\cdot)}(\ell_{q(\cdot)})$ can be easily defined through the quasi-norm

$$\|(f_\nu)_\nu \,|L_{p(\cdot)}(\ell_{q(\cdot)})\| := \Big\| \|(f_\nu(x))_\nu \,|\ell_{q(x)}\| \,|L_{p(\cdot)}\Big\|$$

on sequences $(f_\nu)_{\nu\in\mathbb{N}_0}$ of complex (or extended real-valued) measurable function on $\mathbb{R}^n$.
(Diening, Hästö & Roudenko '09)

The functional $\|(f_\nu)_\nu \,|L_{p(\cdot)}(\ell_{q(\cdot)})\|$ defines always a norm when $\min\{p(x), q(x)\} \geq 1$.

Variable exponent mixed sequence spaces (cont.)

(A. & Hästö '10)
For $p, q \in \mathcal{P}(\mathbb{R}^n)$, the **mixed sequence-Lebesgue space** $\ell_{q(\cdot)}(L_{p(\cdot)})$ consists of all sequences $(f_\nu)_{\nu\in\mathbb{N}_0}$ of complex (or extended real-valued) functions such that $\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}\big(\frac{1}{\mu}(f_\nu)_\nu\big) < \infty$ for some $\mu > 0$, where

$$\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}\big((f_\nu)_\nu\big) := \sum_{\nu\geq 0} \inf\Big\{\lambda_\nu > 0 : \ \varrho_{p(\cdot)}\Big(\frac{f_\nu}{\lambda_\nu^{1/q(\cdot)}}\Big) \leq 1\Big\}.$$

The functional

$$\|(f_\nu)_\nu \,|\ell_{q(\cdot)}(L_{p(\cdot)})\| := \inf\Big\{\mu > 0 : \ \varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}\big(\tfrac{1}{\mu}(f_\nu)_\nu\big) \leq 1\Big\}$$

defines a quasi-norm in $\ell_{q(\cdot)}(L_{p(\cdot)})$ for every $p, q \in \mathcal{P}(\mathbb{R}^n)$.

Besov and Triebel-Lizorkin spaces

Definition

Let $\boldsymbol{w} = (w_j)_j \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ and $p, q \in \mathcal{P}(\mathbb{R}^n)$.

(i) $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$ is the set of all $f \in \mathcal{S}'$ such that

$$\|f\,|B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}\| := \left\|(w_j(\varphi_j * f))_j\,|\,\ell_{q(\cdot)}(L_{p(\cdot)})\right\| < \infty.$$

(ii) $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$ is the set of all $f \in \mathcal{S}'$ such that

$$\|f\,|F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}\| := \left\|(w_j(\varphi_j * f))_j\,|\,L_{p(\cdot)}(\ell_{q(\cdot)})\right\| < \infty$$

(with $p^+, q^+ < \infty$).

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$$\|f \,|\, F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}\| := \big\|(w_j(\varphi_j * f))_j \,|\, L_{p(\cdot)}(\ell_{q(\cdot)})\big\| < \infty$$

(with $p^+, q^+ < \infty$).

The basis system...

Let (φ, Φ) be a pair of functions in $\mathcal{S}$ such that

- $\operatorname{supp} \hat{\varphi} \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\}$ and $|\hat{\varphi}(\xi)| \geq c > 0$ when $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$;
- $\operatorname{supp} \hat{\Phi} \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2\}$ and $|\hat{\Phi}(\xi)| \geq c > 0$ when $|\xi| \leq \frac{5}{3}$.

Setting $\varphi_j := 2^{jn}\varphi(2^j \cdot)$ for $j \in \mathbb{N}$ and $\varphi_0 := \Phi$, then $\varphi_j \in \mathcal{S}$ and

$$\operatorname{supp} \widehat{\varphi}_j \subset \{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}, \quad j \in \mathbb{N}.$$

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$$\operatorname{supp} \widehat{\varphi_j} \subset \{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}, \quad j \in \mathbb{N}.$$

If $w \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ and $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$ (with $\max\{p^+, q^+\} < \infty$ in the F-case), then the spaces $B^{w}_{p(\cdot),q(\cdot)}$ and $F^{w}_{p(\cdot),q(\cdot)}$ are independent of the admissible pair (φ, Φ) taken in its definition, in the sense that different such pairs produce equivalent quasi-norms.

Besov and Triebel-Lizorkin spaces

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(i) $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$ is the set of all $f \in \mathcal{S}'$ such that

$$\|f \,|\, B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}\| := \left\|(w_j(\varphi_j * f))_j \,|\, \ell_{q(\cdot)}(L_{p(\cdot)})\right\| < \infty.$$

(ii) $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$ is the set of all $f \in \mathcal{S}'$ such that

$$\|f \,|\, F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}\| := \left\|(w_j(\varphi_j * f))_j \,|\, L_{p(\cdot)}(\ell_{q(\cdot)})\right\| < \infty$$

(with $p^+, q^+ < \infty$).

The admissible weight sequences...

Definition

Let $\alpha, \alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha \geq 0$ and $\alpha_1 \leq \alpha_2$. We say that a sequence of positive measurable functions $\boldsymbol{w} = (w_j)_j$ belongs to class $\mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ if

(i) there exists $c > 0$ such that

$$0 < w_j(x) \leq c\, w_j(y) \left(1 + 2^j|x-y|\right)^{\alpha}$$

for all $j \in \mathbb{N}_0$ and $x, y \in \mathbb{R}^n$;

(ii) there holds

$$2^{j\alpha_1}\, w_j(x) \leq w_{j+1}(x) \leq 2^{j\alpha_2}\, w_j(x)$$

for all $j \in \mathbb{N}_0$ and $x, y \in \mathbb{R}^n$.

Examples

The scales $A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$, $A \in \{B, F\}$, include various known spaces. (Kempka '09)

- 2−**microlocal spaces** (Peetre'75, Bony'84, Jaffard & Meyer'96) $B^{\boldsymbol{w}}_{2,2} = H^{s,s'}_{x_0}$ and $B^{\boldsymbol{w}}_{\infty,\infty} = C^{s,s'}_{x_0}$:

$$w_j(x) = 2^{js}(1 + 2^j |x - x_0|)^{s'} \qquad (s, s' \in \mathbb{R}, \ \ x_0 \in \mathbb{R}^n).$$

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- **Spaces of variable smoothness and integrability:** $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)} = B^{s(\cdot)}_{p(\cdot),q(\cdot)}$ and $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)} = F^{s(\cdot)}_{p(\cdot),q(\cdot)}$ are the scales introduced by A. & Hästö'10 and Diening, Hästö & Roudenko'09, resp., with

$$w_j(x) = 2^{js(x)}, \qquad s(\cdot) \text{ locally log-Hölder cont.}$$

Examples (cont.)

- Taking $\boldsymbol{w} = (\sigma_j)_j$ with $(\sigma_j)_j$ a sequence of nonnegative real numbers satisfying

$$d_1 \sigma_j \leq \sigma_{j+1} \leq d_2 \, \sigma_j, \quad j \in \mathbb{N}_0,$$

  for some $d_1, d_2 > 0$ (independent of j), then $A^{\boldsymbol{w}}_{p,q} = A^{\sigma}_{p,q}$ are **spaces of generalized smoothness** studied by Farkas & Leopold '06; Moura '01; Kalyabin & Likorkin '87, Merucci'83, ...
- Taking $w_j(x) = 2^{js} \rho(x)$, with $s \in \mathbb{R}$ and $\rho(x)$ satisfying

$$0 < \rho(x) \lesssim \rho(y)(1 + |x - y|^2)^{\frac{\beta}{2}}, \quad x, y \in \mathbb{R}^n \quad (\beta \geq 0),$$

  one obtains **weighted spaces** $A^{\boldsymbol{w}}_{p,q} = A^{s}_{p,q}(\mathbb{R}^n, \rho)$. (Edmunds & Triebel '96)

Variable integrability

2-microlocal spaces

Atomic / molecular decompositions

Sobolev embeddings

Convergence / Applications

Atoms

Definition

Let $K, L \in \mathbb{N}_0$ and $d > 1$. For each $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, a C^K-function a_{jm} on $\mathbb{R}^n$ is called a (K, L, d)-atom (supported near Q_{jm}) if

$$\operatorname{supp} a_{jm} \subset d\, Q_{jm}\,,$$

$$\sup_{x\in\mathbb{R}^n} |D^\gamma a_{jm}(x)| \leq 2^{|\gamma| j}\,, \quad 0 \leq |\gamma| \leq K,$$

and

$$\int_{\mathbb{R}^n} x^\gamma\, a_{jm}(x)\, dx = 0\,, \quad 0 \leq |\gamma| < L.$$

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Remark:
- $L = 0$: no moment conditions;
- Q_{jm}, with $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$: the closed cube with sides parallel to the coordinate axes, centred at $2^{-j}m$ and with side length 2^{-j}.

Molecules

Definition

Let $K, L \in \mathbb{N}_0$ and $M > 0$. For each $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, a C^K-function $[aa]_{jm}$ on $\mathbb{R}^n$ is called a (K, L, M)-molecule (concentrated near Q_{jm}) if

$$|D^\gamma (aa)_{jm}(x)| \leq 2^{|\gamma| j}(1+2^j|x-2^{-j}m|)^{-M}\,, \quad x \in \mathbb{R}^n,\; 0 \leq |\gamma| \leq K,$$

and

$$\int_{\mathbb{R}^n} x^\gamma\, [aa]_{jm}(x)\, dx = 0\,, \quad 0 \leq |\gamma| < L.$$

Lemma ($\approx$ Kempka '10)

Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$*,* $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$*. Let also* $K, L \in \mathbb{N}_0$ *and* $M > 0$ *such that*

$$L > \sigma_{p^-} - \alpha_1 \quad \text{and} \quad M > L + 2n + 2\alpha + 2\sigma_{p^-}\, c_{\log}(1/p).$$

Let $[aa]_{0m}$*,* $m \in \mathbb{Z}^n$*, be* $(K, 0, M)$*-molecules and, for* $j \in \mathbb{N}$ *and* $m \in \mathbb{Z}^n$*,* $[aa]_{jm}$ *be* (K, L, M)*-molecules. If* $\lambda = (\lambda_{jm}) \in b^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$*, then*

$$\sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}\, [aa]_{jm} \quad \text{converges in } \mathcal{S}'.$$

Notation: $\sigma_r := n\left(\frac{1}{\min\{1,r\}} - 1\right)$ and $\sigma_{r\,t} := n\left(\frac{1}{\min\{1,r,t\}} - 1\right)$.

Remark: the proof uses a discrete Sobolev embedding (later...)

Sequence spaces

Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ and $p, q \in \mathcal{P}(\mathbb{R}^n)$. The sets $b^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$ and $f^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$ consist of all (complex-valued) sequences $\lambda = (\lambda_{jm})_{\substack{j\in\mathbb{N}_0\\ m\in\mathbb{Z}^n}}$ such that

$$\|\lambda\,|b^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}\| := \left\| \left(\sum_{m\in\mathbb{Z}^n} \lambda_{jm}\, w_j(2^{-j}m)\,\chi_{jm} \right)_j |\ell_{q(\cdot)}(L_{p(\cdot)}) \right\| < \infty$$

and

$$\|\lambda\,|f^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}\| := \left\| \left(\sum_{m\in\mathbb{Z}^n} \lambda_{jm}\, w_j(2^{-j}m)\,\chi_{jm} \right)_j |L_{p(\cdot)}(\ell_{q(\cdot)}) \right\| < \infty,$$

respectively.

Theorem (Atomic decomposition)

Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ and $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$ (with $p^+, q^+ < \infty$ in the case F-case). Let $K, L \in \mathbb{N}_0$ and $d > 1$ be such that

$$K > \alpha_2 \quad \text{and} \quad L > \sigma_{p^-} - \alpha_1 + c_{\log}(1/q) \quad \textit{(in the B-case);}$$

$$K > \alpha_2 \quad \text{and} \quad L > \sigma_{p^- q^-} - \alpha_1 \quad \textit{(in the F-case).}$$

Then $f \in \mathcal{S}'$ belongs to $A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$ ***if and only if*** *there exist $\lambda \in a^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$ and $(K, 0, d)$-atoms a_{0m}, $m \in \mathbb{Z}^n$, and (K, L, d)-atoms a_{jm}, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, such that*

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}\, a_{jm} \quad \textit{(convergence in } \mathcal{S}'\textit{).} \tag{3.1}$$

Moreover,

$$\inf \|\lambda \,|\, a^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}\|$$

defines a quasinorm in $A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$ which is equivalent to $\|\cdot\,|A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}\|$, where the infimum runs over all $\lambda \in a^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$ that can be used in (3.1) (for fixed $f \in A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$) for all possible atoms with the properties above.

Theorem (Molecular decomposition)

Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ and $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$ (with $p^+, q^+ < \infty$ in the case F-case). Let $K, L \in \mathbb{N}_0$ and $M > 0$ s.t.

$$K > \alpha_2\,, \quad L > \sigma_{p^-} - \alpha_1 + c_{\log}(1/q)\,,$$

$$M > L + 2n + 2\alpha + \sigma_{p^-} \max\{1, 2\,c_{\log}(1/p)\} + c_{\log}(1/q) \quad \text{in the B-case, or}$$

$$K > \alpha_2\,, \quad L > \sigma_{p^-q^-} - \alpha_1\,, \quad M > L + 2n + 2\alpha + \sigma_{p^-q^-} \max\{1, 2\,c_{\log}(1/p)\},$$

in the F-case. Then $f \in \mathcal{S}'$ belongs to $A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$ ***if and only if*** *there exist $\lambda \in a^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$ and $(K, 0, M)$-molecules $[aa]_{0m}$, $m \in \mathbb{Z}^n$, and (K, L, M)-molecules $[aa]_{jm}$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, such that*

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}\,[aa]_{jm} \quad \text{(convergence in } \mathcal{S}'\text{)}. \qquad (3.2)$$

Moreover,

$$\inf \|\lambda \,|\, a^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}\|$$

defines a quasinorm in $A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$ which is equivalent to $\|\cdot\,|A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}\|$, where the infimum runs over all $\lambda \in a^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$ that can be used in (3.2) (for fixed $f \in A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$) for all possible molecules of the type above.

Remarks

- The atomic / molecular representations above complement results obtained by Kempka '10;
  variable exponents q are included in the B case now.
- Atomic decompositions have been used to get other fundamental properties: Sobolev embeddings, traces (Gonçalves, Moura, Neves '14)
- Further applications: in the sequel...

Sobolev embedding

Theorem

Let $\mathbf{w^1} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $p_0, p_1 \in \mathcal{P}^{\log}(\mathbb{R}^n)$ *and* $\frac{1}{q}$ *be locally* log*-Hölder continuous. If* $p_0 \leq p_1$ *and*

$$\frac{w_j^0(x)}{w_j^1(x)} = 2^{j\left(\frac{n}{p_0(x)} - \frac{n}{p_1(x)}\right)}, \quad x \in \mathbb{R}^n, \quad j \in \mathbb{N}_0,$$

then

$$B^{\mathbf{w^0}}_{p_0(\cdot),q(\cdot)} \hookrightarrow B^{\mathbf{w^1}}_{p_1(\cdot),q(\cdot)}.$$

Proof (idea):

- First we show the discrete analogue

$$b^{\mathbf{w}^0}_{p_0(\cdot),q(\cdot)} \hookrightarrow b^{\mathbf{w}^1}_{p_1(\cdot),q(\cdot)}.$$

(hard to obtain for variable q).

- Then transfer the result to B-spaces via atomic decompositions.

From the previous theorem, we can easily show that

$$\mathcal{S} \hookrightarrow A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)} \hookrightarrow \mathcal{S}'$$

for $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ and $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$ ($\max\{p^+, q^+\} < \infty$ in the F case).

Theorem

Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $K, L \in \mathbb{N}_0$ and $M > 0$. Let $[aa]_{0m}$, $m \in \mathbb{Z}^n$, be $(K, 0, M)$-molecules and, for $j \in \mathbb{N}$ and $m \in \mathbb{Z}^n$, $[aa]_{jm}$ be (K, L, M)-molecules. If $\lambda = (\lambda_{jm}) \in a^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$ and

$$K > \alpha_2\,, \quad L > \sigma_{p^-} - \alpha_1 + c_{\log}(1/q)\,,$$

$$M > L+2n+2\alpha+\sigma_{p^-} \max\{1, 2\,c_{\log}(1/p)\} + c_{\log}(1/q) \quad \text{when } a = b, \text{ or}$$

$$K > \alpha_2\,, \quad L > \sigma_{p^- q^-} - \alpha_1\,, \quad M > L+2n+2\alpha+\sigma_{p^- q^-} \max\{1, 2\,c_{\log}(1/p)\},$$

when $a = f$ (with $p^+, q^+ < \infty$ in this case), then

$$g := \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}\,[aa]_{jm} \in A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}.$$

Moreover, the convergence to g holds also in $A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$ under the additional assumptions $p^+, q^+ < \infty$.

Remarks:

- In the sum above, the convergence holds in $A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$ with respect to both sums.
- Moreover, we can even show that

$$g = \sum_{(\nu,m)\in\mathbb{N}_0\times\mathbb{Z}^n} \lambda_{\nu m}\,[aa]_{\nu m}$$

with summability in $A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$.

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Corollary

If $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ and $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $\max\{p^+, q^+\} < \infty$, then $\mathcal{S}$ is dense in $A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$.

The proof...

(i) Consider $f \in A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$. Using atomic decompositions results, we may write

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}\, a_{\nu m} \quad \text{(convergence in } \mathcal{S}'\text{)}$$

for (K, L, d)-atoms $a_{\nu m}$ in $\mathcal{S}$, where $K, L \in \mathbb{N}_0$ and $d > 1$ are at our disposal.

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for (K, L, d)-atoms $a_{\nu m}$ in $\mathcal{S}$, where $K, L \in \mathbb{N}_0$ and $d > 1$ are at our disposal.

(ii) Choosing K, L large enough, we know that

$$f = \sum_{(\nu,m)\in\mathbb{N}_0\times\mathbb{Z}^n} \lambda_{\nu m}\, a_{\nu m} \quad \text{with summability in } A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}.$$

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(i) Consider $f \in A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$. Using atomic decompositions results, we may write

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\, a_{\nu m} \quad \text{(convergence in } \mathcal{S}'\text{)}$$

for (K, L, d)-atoms $a_{\nu m}$ in $\mathcal{S}$, where $K, L \in \mathbb{N}_0$ and $d > 1$ are at our disposal.

(ii) Choosing K, L large enough, we know that

$$f = \sum_{(\nu,m)\in\mathbb{N}_0\times\mathbb{Z}^n} \lambda_{\nu m}\, a_{\nu m} \quad \text{with summability in } A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}.$$

(iii) Since $\mathbb{N}_0 \times \mathbb{Z}^n$ is infinitely countable,

$$f = \sum_{j=1}^{\infty} \lambda_j\, a_j = \lim_{J\to\infty} \sum_{j=1}^{J} \lambda_j\, a_j \quad \text{(convergence in } A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}\text{)}.$$

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