Hard Lefschetz Theorem for Sasakian manifolds

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Sasakian manifolds

Let (M^{2n+1}, g) be a Riemannian manifold, η a 1-form, such that

$$\eta \wedge (\mathrm{d}\eta)^n \quad \text{is a volume form.}$$

Then, (M^{2n+1}, η, g) is a Sasakian manifold if and only if

$$(M^{2n+1} \times \mathbb{R}_+, \omega = d(r^2\eta), G = r^2 g + dr^2)$$

is a Kähler manifold.

Hard Lefschetz Theorem for Kähler manifolds

Theorem (Lefschetz 1924, Hodge 1952)

Let (M^{2n}, ω, g) be a compact Kähler manifold. Then, for each $p \leq n$ the map

$$\omega^p \wedge - \colon \Omega_\Delta^{n-p}(M) \to \Omega_\Delta^{n+p}(M)$$
$$\alpha \mapsto \omega^p \wedge \alpha$$

is an isomorphism.

Note that the map $\omega \wedge -$ sends harmonic forms to harmonic forms.

Hard Lefschetz Theorem for Sasakian manifolds

In a compact Sasakian manifold (M^{2n+1}, η, g) one would like to define

$$\eta \wedge (d\eta)^p \wedge - : \Omega_\Delta^{n-p}(M) \to \Omega_\Delta^{n+p+1}(M)$$
$$\alpha \mapsto \eta \wedge (d\eta)^p \wedge \alpha$$

and to get isomorphisms.

PROBLEM: Neither $d\eta \wedge -$ nor $\eta \wedge d\eta \wedge -$ send harmonic forms into harmonic forms! So, a priori the above maps are not well defined.

However, the claim turns out to be true. So, what happens?

Important subspaces

$$\alpha \in \Omega^{p,\lambda}_{\blacksquare}(M) \overset{def}{\iff} \begin{cases} \boxed{\Delta\alpha = \lambda\alpha} \\ d\alpha = 0 \\ i_\xi \alpha = 0 \\ \eta \wedge \delta\alpha = 0 \end{cases}$$

$$\alpha \in \Omega^{p,\lambda}_{\bullet}(M) \overset{def}{\iff} \begin{cases} \boxed{\Delta\alpha = \lambda\alpha} \\ \delta\alpha = 0 \\ \eta \wedge \alpha = 0 \\ i_\xi d\alpha = 0 \end{cases}$$

Harmonic p-forms

By definition,

$$\Omega^{p,0}_{\blacksquare}(M) \subset \Omega^{p}_{\Delta}(M)$$

On the other hand, for $p \leq n$, every harmonic p-form belongs to $\Omega^{p,0}_{\blacksquare}(M)$ since $d\alpha = 0$, $\delta\alpha = 0$, and [Tachibana]

$$i_\xi \alpha = 0.$$

Thus,

Property

Let M be a compact Sasakian manifold of dimension $2n + 1$. For $p \leq n$,

$$\Omega^{p,0}_{\blacksquare}(M) = \Omega^{p}_{\Delta}(M).$$

Moreover, $\Omega^{p,0}_{\bullet}(M) = 0$.

Harmonic p-forms

Property

For $p \geq n + 1$,

$$\Omega^{p,0}_{\bullet}(M) = \Omega^{p}_{\Delta}(M)$$

Moreover, $\Omega^{p,0}_{\blacksquare}(M) = 0$.

Some information on the spectrum of Δ

Theorem

Let M be a compact Sasakian manifold. We have the pair of inverse isomorphisms

$$\Omega_{\blacksquare}^{p,4\nu}(M) \underset{i_\xi}{\overset{\eta\wedge -}{\rightleftarrows}} \Omega_{\bullet}^{p+1,4(\nu-p+n)}(M) \ . \tag{1}$$

Proposition

Let M be a compact Sasakian manifold and $\nu \neq 0$. We have the pair of isomorphisms

$$\Omega_{\bullet}^{p,4\nu}(M) \underset{\delta}{\overset{d}{\rightleftarrows}} \Omega_{\blacksquare}^{p+1,4\nu}(M) \ , \tag{2}$$

for any $0 \leq p \leq 2n$.

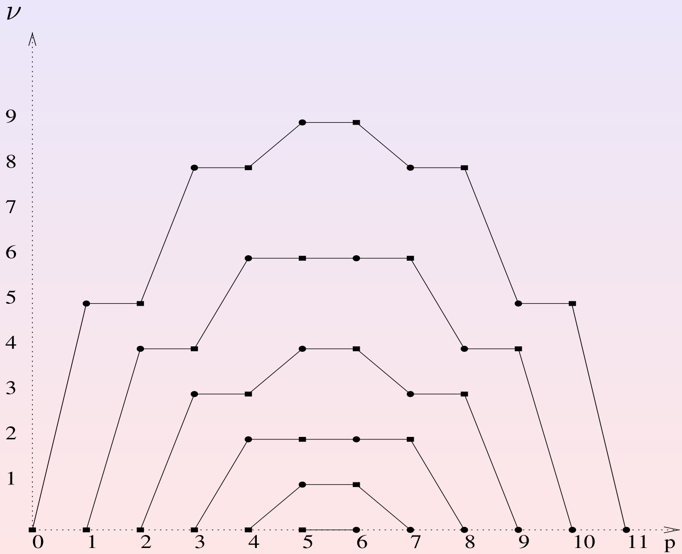
Some information on the spectrum of Δ

Putting together the two isomorphisms (**??**) and (**??**), we have

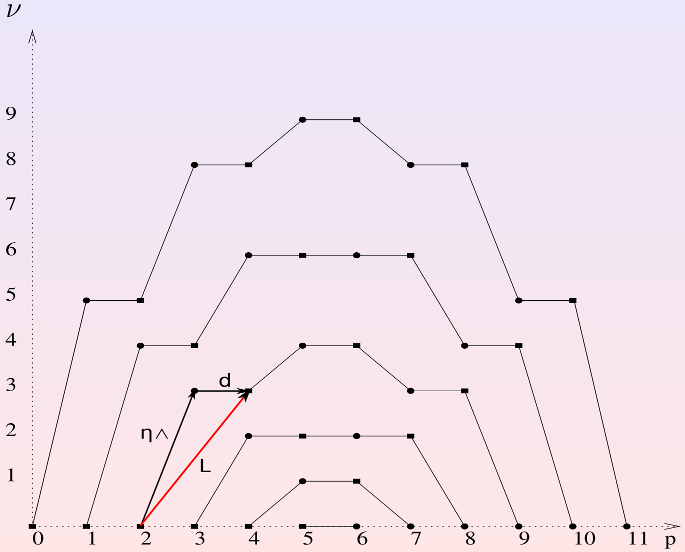
$$
\begin{tikzcd}
\Omega^{p,4\nu}_{\blacksquare}(M) \arrow[r, shift left, "\eta\wedge-"] \arrow[rr, bend left, "L"] & \Omega^{p+1,4(\nu-p+n)}_{\bullet}(M) \arrow[l, shift left, "i_\xi"] \arrow[r, shift left, "d"] & \Omega^{p+2,4(\nu-p+n)}_{\blacksquare}(M). \arrow[l, shift left, "\delta"] \arrow[ll, bend left, "\Lambda"]
\end{tikzcd}
$$

This shows that $L = (d\eta) \wedge -$ and its adjoint Λ induce inverse isomorphisms between the spaces in the diagram.

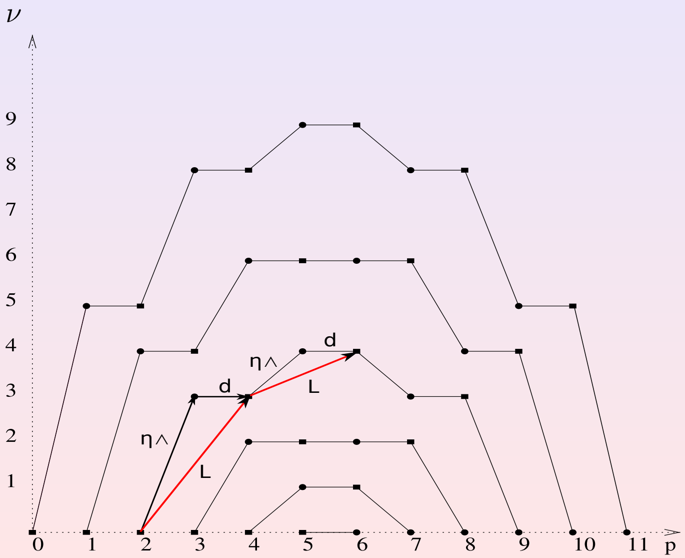
HLT for Sasakian manifolds (n=5)



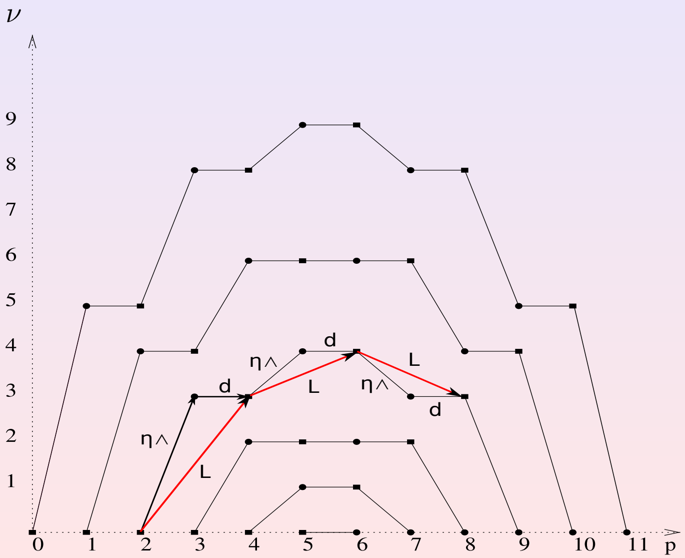
HLT for Sasakian manifolds (n=5)



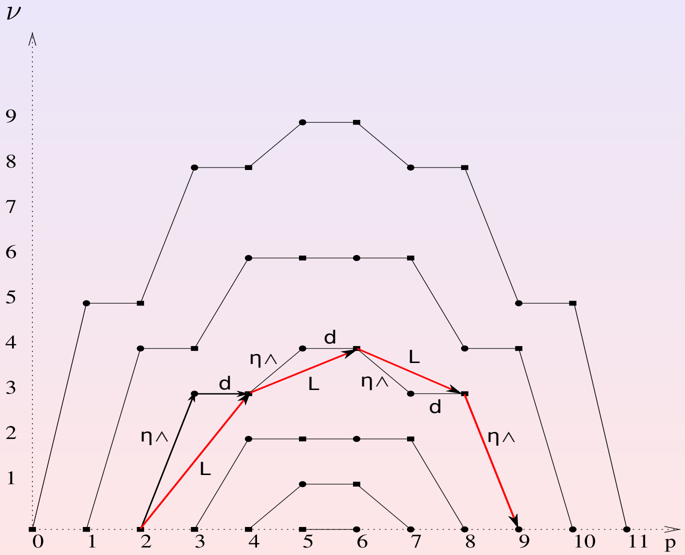
HLT for Sasakian manifolds (n=5)



HLT for Sasakian manifolds (n=5)



HLT for Sasakian manifolds (n=5)



Hard Lefschetz Theorem for Sasakian manifolds

Theorem

Let M a compact Sasakian manifold of dimension $2n+1$ and $p \leq n$. Then the map

$$\begin{aligned} \Omega^p_\Delta(M) &\longrightarrow \Omega^{2n+1-p}_\Delta(M) \\ \alpha &\longmapsto \eta \wedge (d\eta)^{n-p} \wedge \alpha \end{aligned}$$

is an isomorphism.

Hard Lefschetz Theorem in cohomology

For a compact Sasakian manifold (M^{2n+1}, η, g) a naive guess would be to consider:

$$\begin{aligned} H^p(M) &\longrightarrow H^{2n+1-p}(M) \\ [\,\alpha\,] &\longmapsto [\,\eta \wedge (d\eta)^{n-p} \wedge \alpha\,], \end{aligned}$$

PROBLEM:
α closed does NOT imply that $\eta \wedge (d\eta)^{n-p} \wedge \alpha$ is closed!
SOLUTION?
First take the projection on the harmonic part

$$\begin{aligned} H^p(M) &\longrightarrow H^{2n+1-p}(M) \\ [\,\alpha\,] &\longmapsto [\,\eta \wedge (d\eta)^{n-p} \wedge \mathcal{H}\alpha\,] \end{aligned}$$

NEW PROBLEM: $\mathcal{H}\alpha$ could in general depend on the metric!

Hard Lefschetz Theorem for Sasakian manifolds

Theorem

Let (M^{2n+1}, η, g) be a compact Sasakian manifold and $p \leq n$. Let $\mathcal{H}\colon \Omega^p(M) \to \Omega^p_\Delta(M)$ be the projection on the harmonic part. Then the map

$$\begin{aligned} Lef_p\colon H^p(M) &\longrightarrow H^{2n+1-p}(M) \\ [\,\alpha\,] &\longmapsto [\,\eta \wedge (d\eta)^{n-p} \wedge \mathcal{H}\alpha\,], \end{aligned}$$

is an isomorphism. Furthermore, it does not depend on the choice of the Sasakian metric g on (M^{2n+1}, η). Furthermore, it does not depend on the choice of the Sasakian metric g on (M^{2n+1}, η).

A topological obstruction

Let (M^{2n+1}, η) be a compact contact manifold. We can define a relation between $H^p(M)$ and $H^{2n+1-p}(M)$:

$$\mathcal{R}_{Lef_p} = \left\{ \left([\beta] , [\eta \wedge (d\eta)^{n-p} \wedge \beta] \right) \middle| \begin{array}{ll} \beta \in \Omega^p(M), & d\beta = 0, \\ i_\xi \beta = 0, & (d\eta)^{n-p+1} \wedge \beta = 0 \end{array} \right\}.$$

Now, if (M, η) admits a compatible Sasakian metric, then $\mathcal{R}_{Lef_p}$ is the graph of the isomorphism $Lef_p : H^p(M) \longrightarrow H^{2n+1-p}(M)$.

Definition

We say that (M, η) is a *Lefschetz contact manifold* if for every $p \leq n$ the relation $\mathcal{R}_{Lef_p}$ is the graph of an isomorphism between $H^p(M)$ and $H^{2n+1-p}(M)$.

First applications

Theorem

Let (M^{2n+1}, η, g) be a compact Lefschetz contact manifold. Then for each $0 \leq p \leq n$ there exists a nondegenerate bilinear form

$$B : H^p(M) \times H^p(M) \longrightarrow \mathbb{R}$$

defined by

$$B(x, x') = \int_M Lef_p(x) \smile x'.$$

Moreover, the bilinear form B is skew-symmetric for p odd and symmetric for p even.

Corollary

Let $\left(M^{2n+1}, \eta\right)$ be a compact Lefschetz contact manifold. Then the odd Betti numbers b_{2k+1} are even for $0 \leq 2k+1 \leq n$.

First applications

- In 2014, jointly with J.C. Marrero we found examples of compact non-Lefschetz K-contact manifolds in dim. 5 and 7, with b_{2k+1} even for $0 \leq 2k+1 \leq n$.

- Recently, jointly with J.C. Marrero we found an example of a compact non-Sasakian Lefschetz contact manifold in dim. 5.

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