Seidel's morphism of toric 4-manifolds

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Seidel morphism

(M, ω) closed symplectic manifold

Seidel, Lalonde-McDuff-Polterovich:

$$\mathcal{S} : \pi_1(\mathrm{Ham}(M, \omega)) \to QH_*(M, \omega)^\times$$

where $QH_*(M, \omega)^\times$ is the group of invertible elements of the quantum homology of (M, ω).

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- a 4-dimensional closed symplectic manifold endowed with a toric structure and admitting a NEF almost complex structure;
- its correspondent Delzant polytope P;
- an Hamiltonian action generated by a circle subgroup Λ, with moment map Φ_Λ.

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Theorem (McDuff-Tolman)

The Seidel element associated to Λ is

$$\mathcal{S}(\Lambda) = A \otimes qt^{\Phi_{\max}} + \sum_{B \in H_2^S(M;\mathbb{Z})^{>0}} a_B \otimes q^{1-c_1(B)} t^{\Phi_{\max} - \omega(B)}$$

where $H_2^S(M;\mathbb{Z})^{>0}$ consists of the spherical classes of symplectic area $\omega(B) > 0$, $a_B \in H_*(M;\mathbb{Z})$ denotes the contribution of B.

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Corollary

If there exists an almost complex structure J on M so that (M, J) is Fano (there are no J–holomorphic spheres in M with non-positive Chern number), then all the lower order terms vanish and $\mathcal{S}(\Lambda) = A \otimes qt^{\Phi_{\max}}$.

Goal

- To show that if (M, ω) is a 4-dimensional NEF toric manifold, although there are infinitely many contributions to the Seidel elements associated to the Hamiltonian circle actions of such manifolds, these quantum classes can be expressed by explicit closed formulas.

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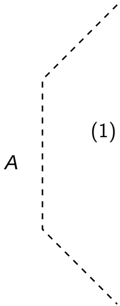
- The formulas only depend on the relative position of representatives of $\pi_2(M)$ with vanishing first Chern number as facets of the moment polytope. In particular, they are directly readable from the polytope.

$\#\{c_1 = 0\} = 0$



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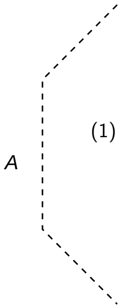

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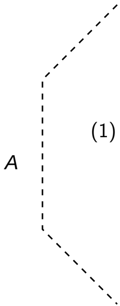

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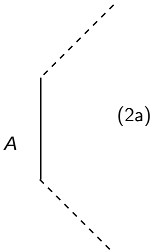

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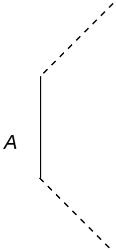
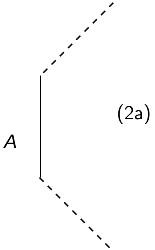


(2a)

kA, with $k > 0$, contributes by $a_{kA} = A$



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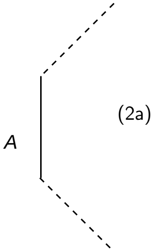

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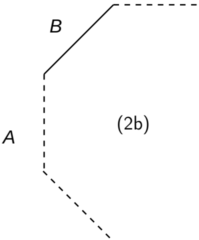

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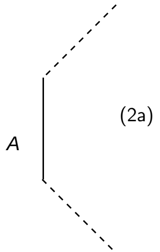

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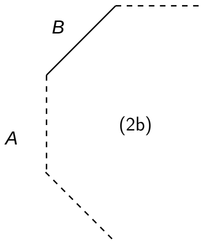

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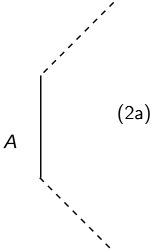

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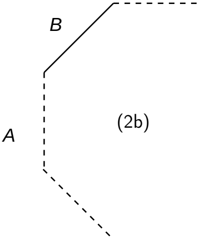

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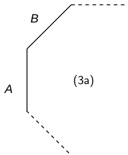

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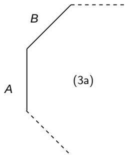

$kA + lB$, with $k \geq 0$ and $l > 0$, contributes by

$$a_{kA+lB} = \begin{cases} A & \text{if } k \geq l \\ -B & \text{if } k < l \end{cases} \implies$$

$$\mathcal{S}(\Lambda) = \left[A \otimes q \, \frac{t^{\Phi_{\max}}}{1-t^{-\omega(A)}} - B \otimes q \, \frac{t^{\Phi_{\max} - \omega(B)}}{1-t^{-\omega(B)}}\right] \cdot \frac{1}{1-t^{-\omega(A)-\omega(B)}}$$

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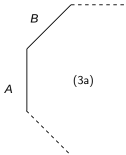

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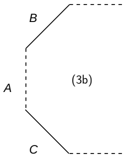
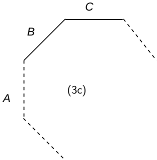


kC and lB, with $k, l \geq 0$, also contribute by

$$a_{kC} = -C \text{ and } a_{lB} = -B \qquad \Longrightarrow$$

$$\mathcal{S}(\Lambda) = A \otimes qt^{\Phi_{\max}} - B \otimes q \frac{t^{\Phi_{\max} - \omega(B)}}{1-t^{-\omega(B)}} - C \otimes q \frac{t^{\Phi_{\max} - \omega(C)}}{1-t^{-\omega(C)}}$$



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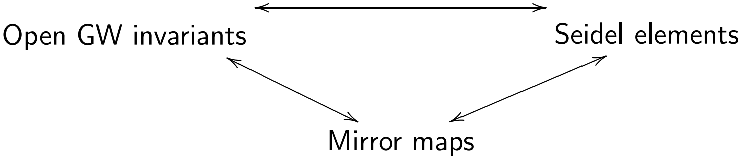
Relation with other works



for toric NEF manifolds.

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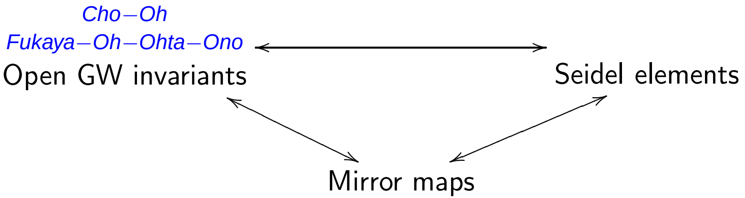

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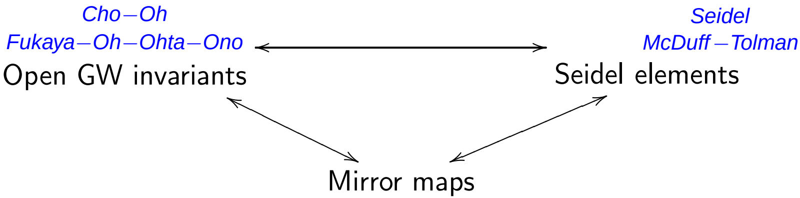

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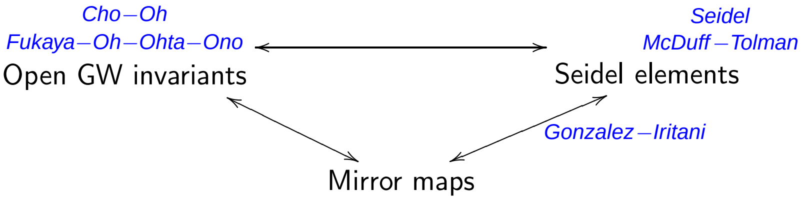

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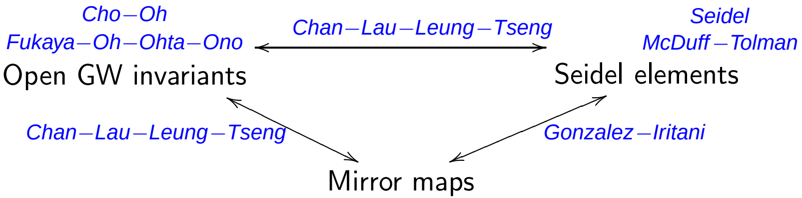

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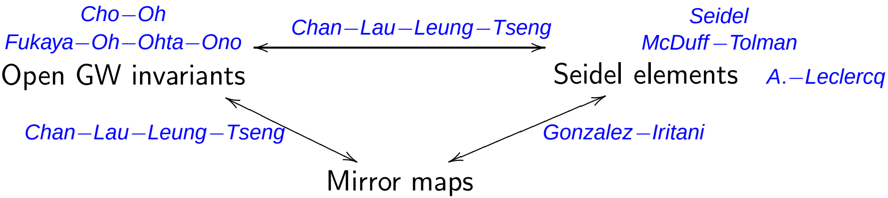


for toric NEF manifolds.

Landau–Ginzburg superpotential

Theorem (Givental)

If (M^{2m}, ω) *is a symplectic NEF manifold, then*

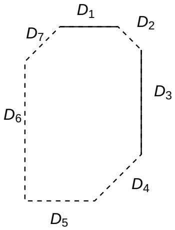
$$QH^*(M,\omega) \cong \check{\Pi}[z_1^{\pm}, \ldots, z_m^{\pm}]/J_W \text{ as } \check{\Pi}\text{–algebras}$$

where $\check{\Pi} := \check{\Pi}^{\mathrm{univ}}[q, q^{-1}]$*, with*

$$\check{\Pi}^{\mathrm{univ}} := \left\{ \sum_{\kappa \in \mathbb{R}} r_\kappa t^\kappa \,\middle|\, r_\kappa \in \mathbb{Q},\ \#\{\kappa < c \mid r_\kappa \neq 0\} < \infty, \forall c \in \mathbb{R} \right\}$$

and J_W *is the ideal generated by all partial derivatives of the superpotential* W.

Example: 4-fold blow-up of the projective plane



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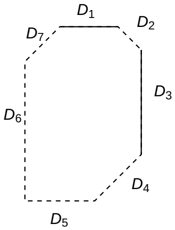

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$$\mathcal{S}(\Lambda_5) = A_5 \otimes qt^{\epsilon_2}, \quad \mathcal{S}(\Lambda_6) = A_6 \otimes qt^{1-\epsilon_1}, \quad \mathcal{S}(\Lambda_1) = A_1 \otimes q\frac{t^{\mu-\epsilon_2}}{1-t^{c_2+c_3-1}},$$

$$\mathcal{S}(\Lambda_3) = A_3 \otimes q\frac{t^{\epsilon_1}}{1-t^{c_1+c_3-\mu}}, \quad \mathcal{S}(\Lambda_4) = A_4 \otimes qt^{\epsilon_1+\epsilon_2-c_1} - A_3 \otimes q\frac{t^{\epsilon_1+\epsilon_2+c_3-\mu}}{1-t^{c_1+c_3-\mu}},$$

$$\mathcal{S}(\Lambda_7) = A_7 \otimes qt^{\mu+1-c_2-\epsilon_1-\epsilon_2} - A_1 \otimes q\frac{t^{\mu+c_3-\epsilon_1-\epsilon_2}}{1-t^{c_2+c_3-1}}$$

$$\mathcal{S}(\Lambda_2) = A_2 \otimes qt^{\mu-c_3+\epsilon_1-\epsilon_2} - A_1 \otimes q\frac{t^{\mu+c_2-1+\epsilon_1-\epsilon_2}}{1-t^{c_2+c_3-1}} - A_3 \otimes q\frac{t^{c_1+\epsilon_1-\epsilon_2}}{1-t^{c_1+c_3-\mu}}.$$

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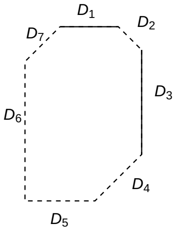

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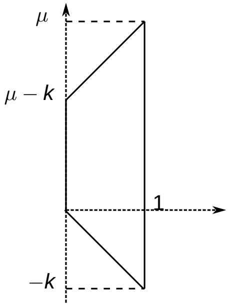
Superpotential: $W = z_2 t^{\mu} + z_1 z_2 t^{\mu-c_3} + z_1 + z_1 z_2^{-1} t^{-c_1} + z_2^{-1} + z_1^{-1} t$

$$z_1^{-1} z_2 t^{\mu+1-c_2} + z_1 t^{\mu-c_1-c_3} + z_2 t^{\mu+1-c_2-c_3}.$$

Non–NEF example



Hirzebruch surfaces:

$$(\mathbb{F}_{2k}, \omega_\mu) \overset{symp}{\sim} (S^2 \times S^2, \mu\omega_0 \oplus \omega_0),$$

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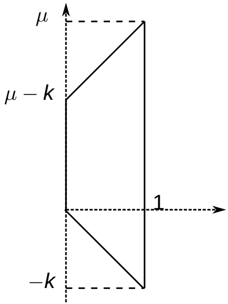

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x_{2k}, y_{2k}: circle actions whose moment maps are, respectively, the first and second components of the moment map associated to the torus action T_{2k} acting on $\mathbb{F}_{2k}$. Denote also by x_{2k}, y_{2k} the generators in $\pi_1(T_{2k})$.

Non-NEF Example

Theorem (Karshon)

$$x_{2k} = kx_2 + (k-1)x_0 \quad \textit{and} \quad y_{2k} = kx_0 + y_0.$$

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Corollary

For the non-NEF toric manifold $\mathbb{F}_4$ we obtain:

$$\mathcal{S}(x_4) = qt^{\frac{1}{2}-2\epsilon} \otimes (B + 2F + B \otimes qt^{1-\mu}) \text{ and } \mathcal{S}(y_4) = F \otimes qt^{\frac{\mu}{2}}.$$

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Remark

If the conjecture by Chan–Lau–Leung–Tseng holds then we obtain the open Gromov-Witten invariants of $\mathbb{F}_4$.

Idea of the proof:

- Construct a Hamiltonian fibration $\pi : (M_\Lambda, \omega_\Lambda) \to (S^2, \omega_0)$ with fiber (M, ω) where $\omega_\Lambda = \Omega + \kappa \pi^*(\omega_0)$ for some big enough κ and Ω is the family (parameterized by S^2) of symplectic forms of the fibers.

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Theorem

If (M^{2m}, ω) is toric then $(M_\Lambda^{2m+2}, \omega_\Lambda)$ is a toric symplectic manifold.

Remark: Also used by González–Iritani and Chan-Lau-Leung-Tseng.

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Reduces the computation of the Seidel elements to the computation of 1-point Gromov-Witten invariants of $(M_\Lambda, \omega_\Lambda)$:

$$a_B \cdot c = \mathrm{GW}^{M_\Lambda}_{A_{\max}+B,1}(c), \quad \text{for all } c \in H_*(M; \mathbb{Z}).$$

where $A_{\max}$ is the homology class of a section.

Recall: $\mathcal{S}(\Lambda) = A \otimes qt^{\Phi_{\max}} + \sum_{B \in H_2^S(M;\mathbb{Z})^{>0}} a_B \otimes q^{1-c_1(B)} t^{\Phi_{\max} - \omega(B)}$.

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- Construct a Hamiltonian fibration $\pi : (M_\Lambda, \omega_\Lambda) \to (S^2, \omega_0)$ with fiber (M, ω) where $\omega_\Lambda = \Omega + \kappa\pi^*(\omega_0)$ for some big enough κ and Ω is the family (parameterized by S^2) of symplectic forms of the fibers.

Theorem

If (M^{2m}, ω) is toric then $(M_\Lambda^{2m+2}, \omega_\Lambda)$ is a toric symplectic manifold.

Reduces the computation of the Seidel elements to the computation of 1-point Gromov-Witten invariants of $(M_\Lambda, \omega_\Lambda)$:

$$a_B \cdot c = \mathrm{GW}^{M_\Lambda}_{A_{\max}+B,1}(c), \quad \text{for all } c \in H_*(M; \mathbb{Z}).$$

where $A_{\max}$ is the homology class of a section.

- Compute these invariants by induction using localization formulas from Spielberg, Liu for the base cases and the splitting axiom satisfied by Gromov-Witten invariants for the inductive steps.