Finite Join Irreducible Semigroups

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Joint research with

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1. Preliminaries

Main reference: “The $\mathfrak{q}$-Theory of Finite Semigroups”, Springer Monographs in Mathematics (2009), by Rhodes & Steinberg

- All semigroups are finite.
- $S \prec T$ means S is a surjective homomorphic image of a subsemigroup of T.
- $S \times T$ denotes the direct product of S and T.
- $S^n = S \times S \times \cdots \times S$ (n times)

1. Preliminaries

- A pseudovariety of finite semigroups is a collection of finite semigroups closed under $\prec$ and $\times$.

- $\mathbf{PV}$ denotes the (complete algebraic) lattice of all pseudovarieties under $\subseteq$.

- (S) denotes the pseudovariety generated by S.

- Pseudovarieties of the form (S) are the compact elements of $\mathbf{PV}$.

1. Preliminaries

An element ℓ of a lattice $(L, \leq)$ is

- join irreducible (ji) if for any $X \subseteq L$,

$$\ell \leq \bigvee X \quad \Longrightarrow \quad \ell \leq x \quad \text{for some } x \in X;$$

- meet irreducible (mi) if for any $X \subseteq L$,

$$\ell \geq \bigwedge X \quad \Longrightarrow \quad \ell \geq x \quad \text{for some } x \in X;$$

- strictly join irreducible (sji) if for any $X \subseteq L$,

$$\ell = \bigvee X \quad \Longrightarrow \quad \ell = x \quad \text{for some } x \in X.$$

2. Definition of Join Irreducible Semigroups

- S is join irreducible (ji) if

$$S \prec A \times B \quad \Longrightarrow \quad (\exists n \geq 1) \quad S \prec A^n \quad \text{or} \quad S \prec B^n.$$

- S is join irreducible (ji) if (S) is join irreducible in $\mathbf{PV}$.

These two definitions are equivalent.

2. Definition of Join Irreducible Semigroups

Stronger definitions

- S is <u>$\times$-prime</u> if

$$S \prec A \times B \quad \Longrightarrow \quad S \prec A \quad \text{or} \quad S \prec B.$$

- S is <u>Kovacs–Newman</u> (KN) if

$$S \overset{f}{\twoheadleftarrow} T \quad \& \quad T \text{ is a subsemigroup of } T_1 \times T_2$$

$$\Longrightarrow \quad f \text{ factors through one of the projections}$$

2. Definition of Join Irreducible Semigroups

Proposition

$$\{\text{KN semigroups}\} \subsetneqq \{\times\text{-prime semigroups}\} \subsetneqq \{\text{ji semigroups}\}$$

- All simple non-Abelian groups are KN.
- Any cyclic group $\mathbb{Z}_p$ of prime order p is $\times$-prime but not KN.
- The Brandt semigroup B_2 of order five is ji but not $\times$-prime.

$$B_2 = \mathcal{M}^0\left(\{1\}, 2, 2; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

3. Three For One (ji/mi Duality)

- If S is ji, then its exclusion class $\text{Excl}(S) = \{T \mid S \not\prec T\}$ is a pseudovariety.

- The pseudovariety Excl(S), being meet irreducible, is defined by a single pseudoidentity.

- Join irreducible pseudovarieties are small.

- Meet irreducible pseudovarieties are big.

4. Operations Preserving Join Irreducibility

(1) S is ji $\Longleftrightarrow$ its dual $\overleftarrow{S}$ is ji.

(2) S is ji $\Longrightarrow$ S^I is ji.

- It is possible that $(S) = (S^I)$.
- Converse of (2) is false.
  - $SL_2 \times RZ_2$ is not ji
  - $\left((SL_2 \times RZ_2)^I\right) = (RZ_2^I) = \mathbf{RRB}$ is ji.

4. Operations Preserving Join Irreducibility

(3) S is ji $\Longrightarrow$ S^{bar} is ji.

- S^{bar} = right faithful action $(S^{\bullet}, S)$ together with all $S^{\bullet}$ constant maps.
- $|S^{\mathsf{bar}}| \leq |S| + |S| + 1$.
- It is possible that $(S) = (S^{\mathsf{bar}})$.
- Converse of (3) is false. Counterexample given later.
- Hence, ignoring trivial cases, if S is a ji semigroup then

$$S, \quad S^{\mathsf{bar}}, \quad \overleftarrow{(S^{\mathsf{bar}})^{\mathsf{bar}}}, \quad \left(\overleftarrow{(S^{\mathsf{bar}})^{\mathsf{bar}}}\right)^{\mathsf{bar}}, \quad \ldots$$

is an infinite list of distinct ji semigroups.

5. Join Irreducible Pseudovarieties Generated by Semigroups of Order Up to 5

Number of semigroups up to isomorphism and anti-isomorphism

n	2	3	4	5	Total
ji semigroups of order n	4	8	33	196	241
semigroups of order n	4	18	126	1160	1308

These semigroups and their dual semigroups generate

30 distinct ji pseudovarieties.

5. Join Irreducible Pseudovarieties Generated by Semigroups of Order Up to 5

(1) $(\mathbb{Z}_2)$, $(\mathbb{Z}_3)$, $(\mathbb{Z}_4)$, $(\mathbb{Z}_5)$, $(\mathbb{Z}_2^{\mathsf{bar}})$, $(\overleftarrow{\mathbb{Z}_2^{\mathsf{bar}}})$

(2) (N_2), (N_3), (N_4), (N_5), (N_1^I), (N_2^I), (N_3^I), (N_4^I)

(N_2^{bar}), $(\overleftarrow{N_2^{\mathsf{bar}}})$, $\left(\{N_2^{\mathsf{bar}}\}^I\right)$, $\left(\overleftarrow{\{N_2^{\mathsf{bar}}\}^I}\right)$,

where $N_k = \langle a \mid a^k = 0 \rangle$.

(3) (LZ_2), (LZ_2^I), (LZ_2^{bar}), (RZ_2), (RZ_2^I), (RZ_2^{bar})

where LZ_2 [RZ_2] is the left [right] zero semigroup of order 2.

5. Join Irreducible Pseudovarieties Generated by Semigroups of Order Up to 5

(4) (A_2), (A_0), (A_0^I)

where $A_2 = \mathcal{M}^0\left(\{1\}, 2, 2; \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)$

and A_0 is a subsemigroup of A_2.

A_0	0	a	b	c
0	0	0	0	0
a	0	0	0	a
b	0	a	b	a
c	0	0	0	c

5. Join Irreducible Pseudovarieties Generated by Semigroups of Order Up to 5

(5) (B_2), (ℓ_3^{bar}), $(\overleftarrow{\ell_3^{\mathsf{bar}}})$

where $B_2 = \mathcal{M}^0\left(\{1\}, 2, 2; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$

and ℓ_3 is a subsemigroup of B_2.

ℓ_3	0	a	b
0	0	0	0
a	0	0	0
b	0	a	b

ℓ_3^{bar}	a	b	c	d	e
a	a	a	a	d	e
b	a	a	a	d	e
c	a	b	c	d	e
d	a	a	a	d	e
e	a	d	e	d	e

Note: ℓ_3^{bar} is ji but ℓ_3 is not.

6. Beyond Order 5

(1) Comments (time permitting) on:

- Semigroups of order 6.
- Bands.

(2) All finite ji Abelian groups: $\mathbb{Z}_{p^n}$ with p prime and $n = 1, 2, \ldots$

6. Beyond Order 5

(3) Finite groups G in general: G is ji $\Longrightarrow$ G is monolithic

(i.e. G contains a unique minimal normal subgroup $N \neq 1$, called the monolith of G.)

- $N \cong S^n$ for some simple group S.
- G is simple non-Abelian $\overset{\text{KN}}{\Longrightarrow}$ G is ji
- (Bergman 2014) $N \cong \mathbb{Z}_p^n$ and N splits $\Longrightarrow$ G is $\times$-prime

  Hence the symmetric group S_3 is ji
- First unknown group case: $(Q_8) = (D_8)$

  It is not $\times$-prime, but is it ji?

6. Beyond Order 5

(4) Known examples of $\mathcal{J}$-trivial ji semigroups:

- $N_n = \langle a \,|\, a^n = 0 \rangle, \;\; n = 1, 2, \ldots$
- $P_n = \langle e, f \,|\, e^2 = e,\, f^2 = f,\, (ef)^n = 0 \rangle, \;\; n = 1, 2, \ldots$
- $Q_n = \langle e, f \,|\, e^2 = e,\, f^2 = f,\, (ef)^n e = 0 \rangle, \;\; n = 1, 2, \ldots$
- $N_n^I, \;\; P_n^I, \;\; Q_n^I, \;\; n = 1, 2, \ldots$
- Are there other examples?

(5) All ji KN semigroups are known.

See "The $\mathfrak{q}$-Theory of Finite Semigroups"

7. More Milage From Old Results

(1) Use of equational theory for small semigroups.

(2) Techniques from complexity (KR) theory of finite semigroups

$$S \text{ is ji} \quad \& \quad S \prec T \wr A \quad \& \quad T \prec T_1 \times T_2$$

$$\Longrightarrow \quad S \prec T_1 \wr A \quad \text{or} \quad S \prec T_2 \wr A$$