Commutants of Toeplitz operators

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Issam Louhichi

American University of Sharjah
Sharjah, UAE

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Preliminaries

Let $L^2(\mathbb{D}, dA)$ be the space of all square integrable functions on the unit disk $\mathbb{D}$ with respect to the normalized Lebesgue measure $dA = rdr\frac{d\theta}{\pi}$.
The analytic Bergman space, denoted by $L_a^2(\mathbb{D})$, is the closed subspace of $L^2(\mathbb{D}, dA)$ consisting of all analytic functions on $\mathbb{D}$. It is well known that $L_a^2(\mathbb{D})$ is a Hilbert space with the set $\{\sqrt{n+1}\, z^n\}_{n=0}^{\infty}$ as an orthonormal basis. Let P be the orthogonal projection of $L^2(\mathbb{D}, dA)$ onto $L_a^2(\mathbb{D})$.

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Definition

For a bounded function f on $\mathbb{D}$, the Toeplitz operator T_f with symbol f is defined on $L_a^2(\mathbb{D})$ by

$$T_f(u) = P(fu), \text{ for } u \in L_a^2.$$

Preliminaries

It is easy to check the following properties of the Toeplitz operator:

- $T_{\alpha f+\beta} = \alpha T_f + \beta I$, where I is the identity operator.
- $T_f^* = T_{\bar{f}}$.
- If $f \in \mathscr{A}^\infty(\mathbb{D})$, then T_f is simply the multiplication operator with f.

The general problem

Under which conditions is the product (composition) of two Toeplitz operators T_f and T_g commutative i.e., $T_f T_g = T_g T_f$?

Well known results

Theorem (S. Axler & Ž. Čučković)

Suppose that f and g are two bounded harmonic *functions on $\mathbb{D}$. Then $T_f T_g = T_g T_f$ if and only if*

(i) *f and g are both analytic on $\mathbb{D}$, or*

(ii) *$\overline{f}$ and $\overline{g}$ are both analytic on $\mathbb{D}$, or*

(iii) *there exist constants $\alpha, \beta \in \mathbb{C}$ such that $f = \alpha g + \beta$ on $\mathbb{D}$.*

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If f is nonconstant function in $\mathscr{A}^\infty(\mathbb{D})$ and $g \in L^\infty(\mathbb{D}, dA)$ such that $T_f T_g = T_g T_f$, then g is analytic.

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Theorem (Ž. Čučković & N. V. Rao)

Let $f, g \in L^{\infty}(\mathbb{D}, dA)$ such that f is radial i.e., $f(z) = f(|z|)$. If $T_f T_g = T_g T_f$, then g is a radial function.

Quasihomogeneous symbol

Definition

A function f is said to be quasihomogeneous of degree p if it is of the form $e^{ip\theta}\phi$, where p is an integer and ϕ is a radial function. In this case the associated Toeplitz operator T_f is also called quasihomogeneous Toeplitz operator of degree p.

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$$f(re^{i\theta}) = \sum_{k\in\mathbb{Z}} e^{ik\theta} f_k(r),$$

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The Mellin Transform

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Therefore, we have

$$T_{e^{ik\theta}f_k}(z^n) = 2(n+k+1)\widehat{f_k}(2n+k+2)z^{n+k}$$

Holomorphic Weighted Shift (HWS)

To exploit well this observation, we introduce the following definition

Definition

Let F be a holomorphic function in the right-half plane $\{z \in \mathbb{C} | \Re z > 0\}$, we define the HWS operator T_F of symbol F and order p on $L^2_a(\mathbb{D})$ by

$$T_F(z^k) = F(k) z^{k+p}.$$

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The product (composition) of two HWS operators of order respectively p and q is a HWS operator of order $p+q$.

On the commutativity of HWS

Theorem

Let T_F and T_G be two HWS operators of order respectively p and q both positive integers. If $T_F T_G = T_G T_F$, then $T_F^m = cT_G^n$ for some constant c and any positive integer m, n such that $mp = nq$.

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Theorem

Let T_F and T_G be two HWS operators. Suppose that there exist two positive integers m and n such that $T_F^m = T_G^n$. Then $T_F T_G = T_G T_F$.

Consequences

Theorem (Bicommutant)

Let T_F, T_G and T_H be HWS operators of order p, q and s respectively. Suppose that T_F commutes with T_H and T_H commutes with T_G. Then T_F and T_G commute with each other.

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If T_f and T_g are two Toeplitz operators with bounded symbols which commute with a quasihomogeneous Toeplitz operator, then they commute with each other.

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Conjecture

If two Toeplitz operators commute with a third one, none of them being the identity, then they commute with each other.