Generalized Killing Spinors on Riemannian Spin^c Manifolds

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The Dirac operator on $\mathbb{R}^n$

On $\mathcal{C}(\mathbb{R}^n)$, we define the Laplacian $\triangle$

$$\triangle := -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

It is a second order differential operator

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$$D = \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j} \quad \text{where} \quad b_j \in \mathcal{C}(\mathbb{R}^n) \ \text{ or } \ \mathcal{C}(\mathbb{C}^n).$$

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(Clifford multiplication) $\begin{cases} b_j^2 = -1, \\ b_j b_k = -b_k b_j \quad \text{for} \quad k \neq j. \end{cases}$

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Pauli matrices

We let b_j "live" in a space "bigger" than $\mathcal{C}(\mathbb{R}^n)$ or $\mathcal{C}(\mathbb{C}^n)$: The space of complex matrices.

On $\mathbb{R}$:

$$D = \underbrace{i}_{b_1} \frac{\partial}{\partial x}.$$

On $\mathbb{R}^2$:

$$D = \underbrace{\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}}_{b_1} \frac{\partial}{\partial x} + \underbrace{\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}}_{b_2} \frac{\partial}{\partial y}.$$

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$\mathbb{R}^n$ is spinorial

More general, on $\mathbb{R}^n$, the operator D acts on $\psi : \mathbb{R}^n \longrightarrow \mathbb{C}^{2^{[\frac{n}{2}]}}$.

- D is called the Dirac operator.
- The functions ψ are called spinors fields.
- $\mathbb{R}^n$ is said to be spinorial and it carries parallel spinors.
- $\mathbb{C}^{2^{[\frac{n}{2}]}}$ is called the spinor bundle.

How/when can we define the Dirac operator on the sphere $\mathbb{S}^n$? On the torus $\mathbb{T}^n$? On a Riemannian manifold (M^n, g)?

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Spinorial manifolds

On (M^n, g), in order to define the Dirac operator (a spinorial structure) we need

$$\boxed{(\Sigma M, \nabla, < .,. >, \text{“}\bullet\text{”})}$$

- ΣM is a complex vector bundle of rank $2^{[\frac{n}{2}]}$, i.e. $\Sigma_x M = \mathbb{C}^{2^{[\frac{n}{2}]}}$.
- ∇ is a covariant derivative on $\Gamma(\Sigma M)$. For $\psi : M \longrightarrow \Sigma M$, we have

$$\begin{aligned} \nabla_X : \Gamma(\Sigma M) &\longrightarrow \Gamma(\Sigma M) \\ \psi &\longrightarrow \nabla_X \psi. \end{aligned}$$

- $< .,. >$ a scalar product on $\Gamma(\Sigma M)$.
- “$\bullet$” is the Clifford multiplication: For $X \in \Gamma(TM)$ and $\psi \in \Gamma(\Sigma M)$,

$$(X \bullet \psi)_x = \text{ combination of Pauli matrices.}$$

Spin and Spin^c structures

The Dirac operator is then given by

$$\boxed{D = \sum_{j=1}^{n} e_j \bullet \nabla_{e_j}}$$

We can prove the existence of a *complex line bundle* L called the the determinant line bundle.

$$\boxed{L = (\det \Sigma M)^{2^{1-[\frac{n}{2}]}}}$$

L is trivial $\Longrightarrow$ M is Spin.

L is not trivial $\Longrightarrow$ M is Spin^c.

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The Schrödinger Lichnerowicz formula

On a Riemannian spinorial manifold (M^n, g), we have

$$D^2 = \triangle + \begin{cases} \frac{1}{4}\mathrm{Scal} & \text{if } M \text{ is Spin}, \\ \\ \frac{1}{4}\mathrm{Scal} + \frac{i}{2}\Omega\bullet & \text{if } M \text{ is Spin}^c, \end{cases}$$

where $i\Omega$ is the curvature of the determinant line bundle.

The Dirac Operator + Spinors

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Seiberg-Witten Theory

General Relativity + Physics

Twistor Theory

Yamabe Problem

Surfaces Theory

Supersymmetry + Supergravity + Superstring

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Special spinors fields, Spectrum of D, Geometry + Topology

Generalized Killing spinors on Spin manifolds

On a Riemannian Spin manifold (M^n, g), a generalized Killing spinor field $\psi \in \Gamma(\Sigma M)$ is given, for all $X \in \Gamma(TM)$, by

$$\boxed{\nabla_X \psi = \lambda X \bullet \psi}$$

λ is a complex function.

Theorem (Friedrich 90, Wang 89, Bär 93, Baum 89, Hijazi 86)

The function λ is real or pure imaginary.

- $\lambda = 0$ *(ψ is parallel)* $\Longrightarrow$ *M is Ricci-flat.*
- $\lambda \neq 0$ *is a real function* $\Longrightarrow$ $\lambda =$ *const. (ψ is a Killing spinor field)*
- $\lambda \neq 0$ *is a pure imaginary function: λ is a constant or a function.*

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Generalized Killing spinors on Spin^c manifolds

(M^n, g) is a Riemannian Spin^c manifold carrying a generalized Killing spinor field ψ.

$$\boxed{\nabla_X \psi = \lambda X \bullet \psi}$$

λ is a complex function.

Theorem (Moroianu 1997, Herzlich-Moroianu 1999)

- $\lambda = 0$ *(ψ is parallel Spin^c), M is not necessarily Ricci-flat but*

$$M = \underbrace{K}_{\textit{Kähler manifold}} \times \underbrace{S}_{\mathrm{Spin}\ \textit{manifold with a parallel spinor}}.$$

- λ *real: If $n \geq 4$, then λ is const. (ψ is a Killing spinor field). M is not necessarily Einstein.*

Two questions:

(M^n, g) is a Riemannian Spin^c manifold carrying a generalized Killing spinor field ψ of Killing function λ.

- $(\mathbb{Q}_1)$: Is λ also real or pure imaginary as in the Spin case?
- $(\mathbb{Q}_2)$: What can be said about the geometry of (M^n, g) when λ is a pure imaginary function or pure imaginary number?

Answer to $(\mathbb{Q}_1)$

Theorem (with N. Grosse, 2015)

Let (M^n, g) be a Riemannian Spin^c manifold with a generalized Killing spinor of Killing function λ (a complex function). If $n \geq 4$, λ is real or pure imaginary.

Answer to $(\mathbb{Q}_2)$

Let (M^n, g) be a complete connected Riemannian Spin^c manifold with an imaginary Killing spinor of Killing function $\lambda = ib$.

Theorem (with N. Grosse, 2015)

One of the following cases occurs:

- *A Riemannian covering of M is isometric to the warped product $(F^{n-1} \times \mathbb{R}, k(t)^2 h + dt^2)$, where (F^{n-1}, h) is a complete Riemannian Spin^c manifold with a non-zero parallel spinor field and k is a function on t.*
- *$(M^n, g) = (\mathbb{H}^n, (2|b|)^{-1} g_{\mathbb{H}})$ is a rescaled hyperbolic space of constant curvature $-4b^2$.*

Perspectives

- Restriction of imaginary Killing Spin^c spinors to submanifolds $\Longrightarrow$ geometry + topology of submanifolds.
- Boundary value problems (BVP) for the Dirac operator on the boundary of manifolds carrying imaginary Killing spinors.
- Imaginary Killing Spin^c spinors on low dimensional manifolds.

Thank you!