Water waves trapped by thin horizontal cylinders in one- and two-layer fluid

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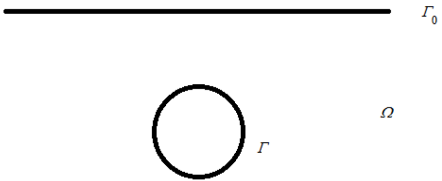
Outline

Ursell's problem

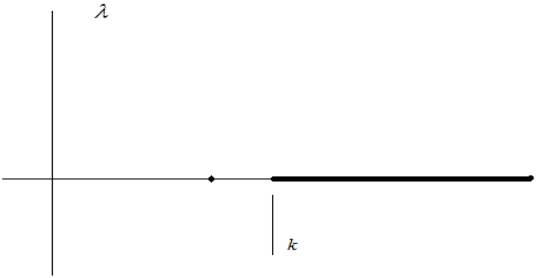


$$\Gamma_0 : \quad \Phi_y = \lambda \Phi, \qquad \Omega : \quad \Delta\Phi - k^2\Phi = 0, \qquad \Gamma : \quad \Phi_n = 0$$

$$\Gamma = \{x = \epsilon \cos t, y = -a + \epsilon \sin t, -\pi \leq t < \pi\}$$

$$e^{i(\omega\tau - kz)}\Phi(x, y), \quad \lambda = \omega^2/g, \quad k > 0 \quad \text{(oblique incidence)}$$

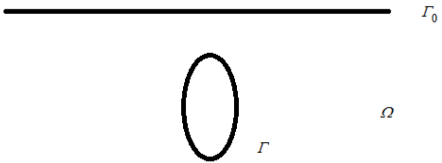
Spectrum



F. Ursell, *Proc. Camb. Phil. Soc.*, 1951, **47**, 347–358

$$\lambda = k(1 - \beta^2), \qquad \beta \sim \epsilon^2$$

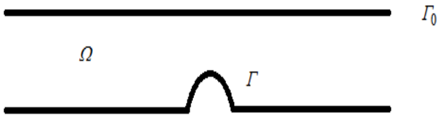
P. McIver



P. McIver, *Q. Jl. Mech. Appl. Math*, 1991, **44**, 193–208

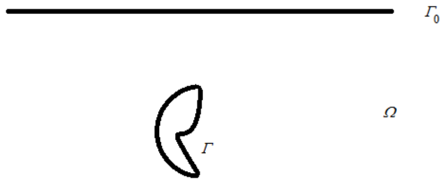
$$\lambda = k(1-\beta^2), \qquad \beta \sim \epsilon^2$$

Underwater ridge



P. Zhevandrov, A. Merzon. *AMS Translations, ser. 2*, 2003, **208**, 235-284
M. I. Romero, P. Zhevandrov. *Russ. J. Math. Phys.*, 2010, **17**, 307-327

$$\lambda = k(\tanh kh_0 - \beta^2), \qquad \beta \sim \epsilon$$



$$\Gamma = \{x = \epsilon X(t), y = -a + \epsilon Y(t), -\pi \leq t < \pi\}, \quad \dot{X}^2 + \dot{Y}^2 \neq 0$$

Dirichlet-Neumann operator

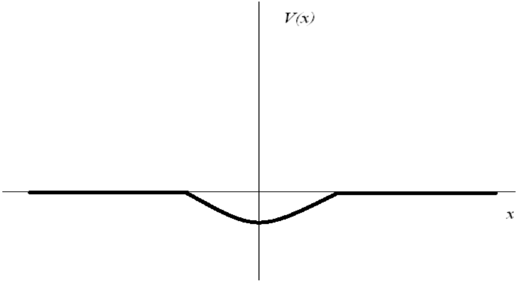
$$\Gamma_0 : \quad \Phi = \varphi, \qquad \Omega : \quad \Delta\Phi - k^2\Phi = 0, \qquad \Gamma : \quad \Phi_n = 0$$

$$\hat{K} : \qquad \Phi|_{\Gamma_0} = \varphi \quad \mapsto \quad \Phi_y|_{\Gamma_0}$$

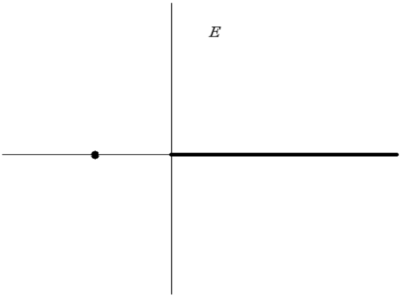
$$\hat{K}\varphi = \lambda\varphi, \qquad \hat{K} = \sqrt{-\partial_x^2 + k^2} + O(\epsilon)$$

Schrödinger Equation

$$\hat{K}_{Sch}\psi = -\psi'' + \epsilon V(x)\psi = E\psi, \qquad \epsilon \ll 1,$$
$$E = -\beta^2, \qquad \psi \simeq \exp(-\beta|x|), \qquad \beta \to 0, \epsilon \to 0$$



Spectrum



The distance between the eigenvalue and the continuous spectrum is β^2.

Fourier transform

$$E = -\beta^2, \qquad \beta > 0.$$

$$-\psi'' + \epsilon V\psi = -\beta^2\psi, \tag{1}$$

Apply Fourier transform to (1):

$$(p^2 + \beta^2)\tilde{\psi}(p) = -\frac{\epsilon}{2\pi}\int \tilde{V}(p - p')\tilde{\psi}(p')dp', \tag{2}$$

Form of the solution

$$\psi \sim e^{-\beta|x|}, \qquad \tilde{\psi}(p) \sim \delta(p),$$
$$\tilde{\psi}(p) = \int e^{-ipx}\psi(x)\,dx = \mathcal{F}_{x\to p}\psi(x)$$

$$(p^2+\beta^2)\tilde{\psi} = -\frac{\epsilon}{2\pi}\int \tilde{V}(p-p')\tilde{\psi}(p')\,dp' \simeq C\tilde{V}(p)$$

$$\tilde{\psi} \sim C\frac{\tilde{V}(p)}{p^2+\beta^2} = \frac{A(p,\epsilon)}{p^2+\beta^2}$$

$$A(p,\epsilon) = A_0(p) + \epsilon A_1(p) + \dots, \qquad \beta = \epsilon\beta_1 + \epsilon^2\beta_2 + \dots$$

Exact solution

Look for the solution of (2) in the form:

$$\tilde{\psi}(p,\epsilon) = \frac{A(p,\epsilon)}{p^2+\beta^2}, \tag{3}$$

Substituting (3) in (2), we obtain:

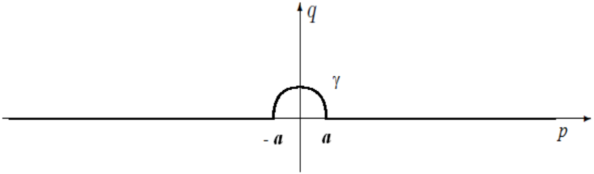
$$A(p,\epsilon) = -\frac{\epsilon}{2\pi}\int \frac{\tilde{V}(p-p')A(p',\epsilon)}{p'^2+\beta^2}dp'. \tag{4}$$

Note that for $\beta = 0$ (4) has a singularity at $p' = 0$.

Exact solution

Introduce

$$\gamma := (-\infty, -1] \cup \{p + iq : p^2 + q^2 = 1, q > 0\} \cup [1, \infty). \quad (5)$$



Exact solution

Apply the Cauchy Residue Theorem to the right-hand side of (4)

$$A(z) = -\frac{\epsilon}{2\pi} \int_{\gamma} \frac{\tilde{V}(z-\zeta)A(\zeta)}{\zeta^2+\beta^2} d\zeta - \frac{\epsilon}{2\beta} \tilde{V}(z-i\beta)A(i\beta), \qquad (6)$$

$\tilde{V}(\zeta)$ is the analytic continuation of $\tilde{V}(p)$ to the complex plane.

Exact solution

Define the integral operator T_β by

$$[T_\beta \varphi(\zeta)](z) = \frac{1}{2\pi} \int_\gamma \frac{\tilde{V}(z-\zeta)\varphi(\zeta)}{\zeta^2 + \beta^2} d\zeta,$$

and write (6) in terms of T_β:

$$[(1 + \epsilon T_\beta) A(\zeta)](z) = -\frac{\epsilon}{2\beta} \tilde{V}(z - i\beta) A(i\beta).$$

Explicit solution

T_β is bounded, ϵT_β is small, for this reason (6) gives

$$A(z) = -\frac{\epsilon}{2\beta} A(i\beta)[(1+\epsilon T_\beta)^{-1}\tilde{V}(\zeta - i\beta)](z), \qquad (7)$$

$(1+\epsilon T_\beta)^{-1} = \sum_{n=0}^{\infty}(-1)^n \epsilon^n T_\beta^n$, $T_\beta^0 \equiv 1$ (the Neumann series).

Explicit solution

Evaluate (7) at $z = i\beta$, multiply by β equation (7) and divide by $A(i\beta)$. We obtain the ***secular equation*** for β :

$$\beta = -\frac{\epsilon}{2}[(1+\epsilon T_\beta)^{-1}\tilde{V}(\zeta - i\beta)](i\beta) \sim -\frac{\epsilon}{2}\int V(x)\,dx. \qquad (8)$$

Water waves – Reduction to Integral Equations

Fundamental solution: $G(x, y) = -\frac{1}{2\pi} K_0(kr)$, $r = \sqrt{x^2 + y^2}$.

$$\Phi(\xi, \eta) = -\int_{\Gamma \cup \Gamma_0} G(x - \xi, y - \eta) \frac{\partial \Phi(x, y)}{\partial n} \, d\ell + \int_{\Gamma \cup \Gamma_0} \frac{\partial G(x - \xi, y - \eta)}{\partial n} \Phi(x, y) \, d\ell.$$

Reduction to Integral Equations

Introduce $\varphi(x) = \Phi(x,0)$, $\theta(t) = \Phi(\epsilon X(t), -a + \epsilon Y(t))$, and let $(\xi,\eta) \to \Gamma_0, \Gamma$.
We obtain

$$\varphi(x) - \frac{\lambda}{\pi}\int K_0\left(k|x-\xi|\right)\varphi(\xi)\,d\xi$$
$$= -\frac{\epsilon k}{\pi}\int_{-\pi}^{\pi}\frac{\zeta(t,x)}{\sigma(t,x)}K_0'\left(k\sigma(t,x)\right)\theta(t)\,dt,$$

$$\zeta(t,x) = x\dot{Y}(t) - a\dot{X}(t) + \epsilon Y(t)\dot{X}(t) - \epsilon X(t)\dot{Y}(t),$$

$$\sigma(t,x) = \sqrt{(x-\epsilon X(t))^2 + (a-\epsilon Y(t))^2},$$

Reduction to Integral Equations

$$\theta(t) + \frac{\epsilon k}{\pi} \int_{-\pi}^{\pi} \frac{\tau(t,s)}{\rho(t,s)} K_0'\left(\epsilon k \rho(t,s)\right) \theta(s)\, ds$$

$$= \frac{1}{\pi} \int \left\{ \lambda K_0(k\sigma(t,x)) - \frac{a - \epsilon X(t)}{\sigma(t,x)} k K_0'\left(k\sigma(t,x)\right) \right\} \varphi(x)\, dx,$$

$$\tau(t,s) = -\dot{Y}(s)\left(X(s) - X(t)\right) + \dot{X}(s)\left(Y(s) - Y(t)\right),$$

$$\rho(t,s) = \sqrt{(X(t) - X(s))^2 + (Y(t) - Y(s))^2}.$$

Fourier transform

$$\tilde{\varphi}(p) = \int e^{-ipx}\varphi(x)\,dx, \qquad \lambda = k(1-\beta^2).$$

$$\left(1 - \frac{k(1-\beta^2)}{\sqrt{p^2+k^2}}\right)\tilde{\varphi} = -\epsilon\int_{-\pi}^{\pi} M_1(p,t,\epsilon)\theta(t)\,dt,$$

$$\theta(t) + \int_{-\pi}^{\pi} M_2(t,s,\epsilon)\theta(s)\,ds = \int M_3(p,t,\epsilon,\beta)\tilde{\varphi}(p)\,dp,$$

$$M_1(p,t,\epsilon) = e^{-i\epsilon pX(t)-(a-\epsilon Y(t))\sqrt{p^2+k^2}} \left(\frac{ip\dot{Y}(t)}{\sqrt{p^2+k^2}} + \dot{X}(t) \right),$$

$$M_2(t,s,\epsilon) = \frac{\epsilon k}{\pi} \frac{\tau(t,s)}{\rho(t,s)} K_0'(k\epsilon\rho(t,s)),$$

$$M_3(p,t,\epsilon,\beta) = \frac{1}{2\pi} e^{i\,p\,\epsilon X(t)-(a-\epsilon Y(t))\sqrt{p^2+k^2}} \left(1 + \frac{k(1-\beta^2)}{\sqrt{p^2+k^2}} \right).$$

Poles

$$L(p,\beta) = 1 - \frac{k(1-\beta^2)}{\sqrt{p^2+k^2}} = \frac{k}{\sqrt{p^2+k^2}}\left(\frac{p^2}{2k^2} + \beta^2 + O\left(p^4\right)\right)$$

$$L(\pm i p_0(\beta),\beta) = 0, \qquad p_0(\beta) = k\beta\sqrt{2-\beta^2}.$$

$$\text{Therefore} \quad \tilde{\varphi}(p) = \frac{A(p)}{L(p,\beta)}$$

System for θ and A

$$A(p) = -\epsilon \int_{-\pi}^{\pi} M_1(p, t, \epsilon)\theta(t)\, dt,$$

$$\left(1 + \hat{M}_2\right)\theta = \hat{M}_\gamma A + \frac{1}{p_0(\beta)} A(ip_0) f(t, \epsilon, \beta),$$

where

$$f(t, \epsilon, \beta) = 2\pi M_3(ip_0, t, \epsilon, \beta)(k^2 - p_0^2)$$

$$\hat{M}_\gamma A = \int_\gamma \frac{M_3(p, t, \epsilon, \beta)}{L(p, \beta)} A(p)\, dp$$

Limiting form

$$K_0'(r) = -\frac{1}{r} + \frac{1}{2}(1-\gamma)r - \frac{1}{2}r\ln\frac{r}{2} + O\left(r^3 \ln r\right)$$

$$M_2(t,s,\epsilon) = M_2^{(0)}(t,s) + \epsilon^2 M_2^{(1)}(t,s,\epsilon,\epsilon\ln\epsilon) + \epsilon^2 \ln\epsilon M_2^{(2)}(t,s,\epsilon,\epsilon\ln\epsilon)$$

In the leading term we have

$$\begin{aligned} M_2^{(0)}(t,s) &= -\frac{1}{\pi}\frac{\tau(t,s)}{\rho^2(t,s)} \\ &= -2\frac{\partial G_0}{\partial n}\left(X(s)-X(t), Y(s)-Y(t)\right)\sqrt{\dot{X}^2(s)+\dot{Y}^2(s)} \end{aligned}$$

$$G_0(x,y) = \frac{1}{2\pi}\ln r$$

Exterior Neumann problem

Exterior Neumann problem

$$\Delta\psi = 0 \quad \text{in} \quad \Omega_0, \qquad \left.\frac{\partial\psi}{\partial n}\right|_C = F, \qquad |\nabla\psi| \to 0 \quad \text{as} \quad r \to \infty,$$

$$C = \{x = X(t), y = Y(t), -\pi \le t < \pi\}$$

Ω_0 – exterior of C on $\mathbb{R}^2$

$\left(1 + \hat{M}_2^{(0)}\right)$ is invertible

Solution

$$\theta = \left(1 + \hat{M}_2\right)^{-1} \hat{M}_\gamma A + \frac{1}{p_0} \left(1 + \hat{M}_2\right)^{-1} A(ip_0) f.$$

$$A(p) = -\epsilon \hat{M}_1 \left(1 + \hat{M}_2\right)^{-1} \hat{M}_\gamma A - \frac{\epsilon}{p_0} \hat{M}_1 \left(1 + \hat{M}_2\right)^{-1} A(ip_0) f.$$

Denoting $\hat{T} = \hat{M}_1 (1 + \hat{M}_2)^{-1} \hat{M}_\gamma$, we have

$$\left(1 + \epsilon \hat{T}\right) A = -\frac{\epsilon}{p_0(\beta)} \hat{M}_1 \left(1 + \hat{M}_2\right)^{-1} A(ip_0) f$$

$$A(p) = -\frac{\epsilon}{p_0(\beta)} \left(1 + \epsilon \hat{T}\right)^{-1} \hat{M}_1 \left(1 + \hat{M}_2\right)^{-1} A(ip_0) f$$

Secular equation

$$p_0(\beta) = -\epsilon \left[\left(1 + \epsilon \hat{T}\right)^{-1} \hat{M}_1 \left(1 + \hat{M}_2\right)^{-1} f \right] \Big|_{p = ip_0(\beta)},$$

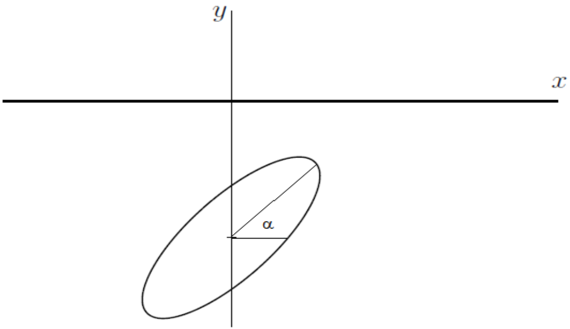
$$\left. \frac{dp_0}{d\beta} \right|_{\beta=0} = k\sqrt{2} \neq 0$$

$$\beta = \frac{1}{\sqrt{2}} \epsilon^2 k^2 e^{-2ak} (S + 2\pi\mu) + O\left(\epsilon^3 \ln \epsilon\right)$$

$$\mu = \frac{1}{2\pi} (S + m_{22})$$

where S is the area inside C, $m_{22} = \int_C n_2 \psi \, dl$ is the added-mass coefficient, ψ is the solution of the exterior Neumann problem with $F = n_2$ (vertical uniform flow past C).

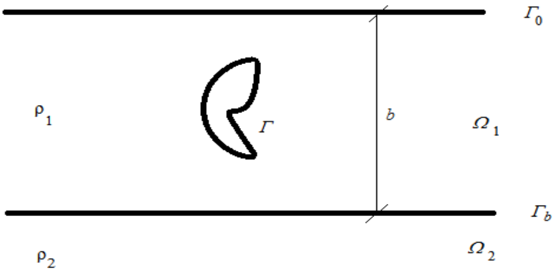
Ellipse



$$X = a_0 \cos(t + \alpha), \quad Y = b_0 \sin(t + \alpha) :$$

$$\beta = \frac{\pi}{\sqrt{2}} k^2 \epsilon^2 e^{-2ak} (a_0^2 \cos^2 \alpha + 2a_0 b_0 + b_0^2 \sin^2 \alpha) + O(\epsilon^3 \ln \epsilon)$$

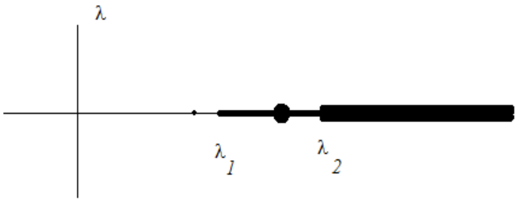
Two-layer fluid



$$\Gamma_0: \quad \Phi_{1y} = \lambda\Phi_1, \quad \Omega_{1,2}: \quad \Delta\Phi_{1,2} - k^2\Phi_{1,2} = 0, \quad \Gamma: \quad \Phi_n = 0$$

$$\Gamma_b: \quad \sigma(\Phi_{1y} - \lambda\Phi_1) = \Phi_{2y} - \lambda\Phi_2, \quad \Phi_{1y} = \Phi_{2y}, \quad \sigma = \rho_1/\rho_2$$

Spectrum two-layer fluid



Continuous spectrum and possible eigenvalues

$$\lambda_1 = k\frac{(1-\sigma)\tanh kb}{1+\sigma\tanh kb}, \quad \lambda_2 = k$$

Discrete eigenvalue

$$\lambda = \lambda_1(1 - \beta^2)$$

$$\beta \sim \epsilon^2 e^{-bk}(A^2 S + B^2 2\pi\mu)$$

$$A = \cosh ak - \frac{\lambda_1}{k} \sinh ak, \qquad B = \sinh ak - \frac{\lambda_1}{k} \cosh ak$$

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