Umbilical properties of co-dimension two submanifolds

José M M Senovilla
Partly in collaboration with N Cipriani

University of the Basque Country, Bilbao, Spain.

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Outline

Introduction

- In General Relativity, timelike and null vector fields play a fundamental role.
- In particular, they describe the (infinitesimal) “evolution” of spacelike submanifolds
- The kinematical quantities (acceleration, expansion, twist and shear) describe different parts of this evolution
- twist: mesaures the rotation. Set it to zero to have orthogonal submanifolds
- acceleration: measures the non-geodecity of the flow lines
- expansion: measures the infinitesimal change of area/volume
- shear: measures the infinitesimal change of “shape”, keeping the area/volume fixed
- *Expansion-free* submanifolds have been deeply studied in the literature (MTS, MOTS) $\rightsquigarrow$ Mean curvature vector field H
  - *shear-free* submanifolds?
- In mathematics they are called umbilical submanifolds
- $\rightsquigarrow$ New vector field G

Plus: a personal Motivation

- A few years ago I presented a complete classification of spacelike surfaces in 4-dimensional Lorentzian manifolds (JMMS, Class. Quantum Grav. 24 (2007) 3091–3124).
- The classification was carried out according to the *extrinsic* properties of the surface: based, at each point, on the properties of two independent Weingarten operators.
- This led to 64 types of points for generic spacelike surfaces.
- To my surprise, this was not enough: the relative orientation of the two Weingarten operators was also needed (at each point).
- The extra parameter can be related to the **commutator of two Weingarten operators**
- We are going to see that this is actually related to the shear, or umbilical, properties of the surface.

Basic concepts and notation

- $(\mathcal{V}, g)$ is an $(n+2)$-dimensional, oriented and time-oriented, Lorentzian manifold $\mathcal{V}$ with metric tensor g of signature $(-,+,\dots,+)$.
- $\forall x \in \mathcal{V}$, the isomorphism between $T_x\mathcal{V}$ and $T_x^*\mathcal{V}$ is denoted by

$$\begin{array}{rcl} \flat : T_x\mathcal{V} & \longrightarrow & T_x^*\mathcal{V} \\ v & \longmapsto & v^\flat \end{array}$$

  and defined by $v^\flat(w) = g(v,w)$, $\forall w \in T_x\mathcal{V}$.
  Its inverse map is denoted by $\sharp$. These maps extend naturally to the tangent and co-tangent bundles.

Definition (Co-dimension two (imbedded) submanifold)

*A **co-dimension two submanifold** —a "surface" if $n = 2$—is (S, Φ), where S is an n-dimensional oriented manifold and $\Phi : S \longrightarrow \mathcal{V}$ is an imbedding.*

(S will be identified with its image $\Phi(S) \subset \mathcal{V}$.)

1st fundamental form and orthogonal splitting

- The **first fundamental form**: $\overline{g} \equiv \Phi^* g$.
- $\overline{g}$ is assumed to be positive definite on S, so that S is spacelike.
- Then, at any $x \in S$ one has the orthogonal decomposition

$$T_x \mathcal{V} = T_x S \oplus T_x S^\perp .$$

- $\mathfrak{X}(S)$ (respectively $\mathfrak{X}(S)^\perp$) will denote the set of smooth vector fields tangent (resp. orthogonal) to S.
- We will often give definitions and properties on $\mathfrak{X}(S)$, but they of course have always a previous, more fundamental, version on each $T_x S$.

Null basis on $\mathfrak{X}(S)^\perp$. Boost freedom

- S having co-dimension two, there are two *independent* normal vector fields on S: two sections on the normal bundle which are linearly independent at each $x \in S$.
- A possible choice is to take two independent normal **null** vector fields (and future-pointing say). We will denote these by $k, \ell \in \mathfrak{X}(S)^\perp$, so that

$$g(\ell,\ell) = g(k,k) = 0, \quad g(\ell,k) = -1$$

- The last of these is a convenient normalization condition
- There remains a boost freedom given simply by

$$\ell \longrightarrow \ell' = e^{\beta}\ell, \qquad k \longrightarrow k' = e^{-\beta}k$$

so that the two independent null *directions* are uniquely determined on S.

Volume forms and Hodge dual operators

- Given the chosen orientations:
  1. ϵ denotes the canonical volume element $(n+2)$-form in $(\mathcal{V}, g)$
  2. $\overline{\epsilon}$ is the corresponding canonical n-form associated to $(S, \overline{g})$
  3. Then, there is an induced volume element 2-form on $\mathfrak{X}(S)^{\perp}$ denoted by $\epsilon^{\perp}$ and defined on S via

$$\epsilon = \epsilon^{\perp} \wedge \overline{\epsilon}$$

- The Hodge dual operator associated to $\epsilon^{\perp}$ is written and defined by

$$\star^{\perp} N \equiv (i_N \epsilon^{\perp})^{\sharp}, \ \ \forall N \in \mathfrak{X}(S)^{\perp}$$

- $\star^{\perp} N$ is well-defined and points along the unique normal direction in $\mathfrak{X}(S)^{\perp}$ orthogonal to the normal $N \in \mathfrak{X}(S)^{\perp}$
- The orientation of the null basis is chosen such that:

$$\epsilon^{\perp} = \ell^{\flat} \wedge k^{\flat} \qquad \Longrightarrow \qquad \star^{\perp} \ell = \ell, \ \ \star^{\perp} k = -k.$$

Covariant derivatives

- Let ∇ denote the canonical connection in $(\mathcal{V}, g)$
- The *Levi-Civita connection* $\overline{\nabla}$ on $(S, \overline{g})$ ($\overline{\nabla}\overline{g} = 0$) can be defined as

$$\overline{\nabla}_X Y \equiv (\nabla_X Y)^T \qquad \forall X, Y \in \mathfrak{X}(S)$$

- The *normal connection* D acts, in turn, on $\mathfrak{X}(S)^{\perp}$

$$D_X : \mathfrak{X}(S)^{\perp} \longrightarrow \mathfrak{X}(S)^{\perp}$$

for $X \in \mathfrak{X}(S)$, and is given by the standard definition

$$D_X N \equiv (\nabla_X N)^{\perp}, \quad \forall N \in \mathfrak{X}(S)^{\perp} \quad \forall X \in \mathfrak{X}(S).$$

The normal connection one-form s

- Given the null basis, a one-form $s \in \Lambda^1(S)$ is defined by

$$s(X) \equiv -g(k, D_X \ell) = g(D_X k, \ell) \qquad \forall X \in \mathfrak{X}(S)$$

- Therefore, for all $X \in \mathfrak{X}(S)$

$$D_X \ell = s(X)\ell, \qquad D_X k = -s(X)k$$

- Observe that s is not invariant under boost rotations $(\ell, k) \to (e^{\beta}\ell, e^{-\beta}k)$.
- Actually, it behaves like a "connection" :

$$s'(X) = s(X) + X(\beta)$$

or equivalently

$$s' = s + d\beta.$$

- Thus, the 2-form ds is invariant and well-defined (we will see it describes basically the **normal curvature**).

Extrinsic geometry

- The basic extrinsic object is the **shape tensor** (also called **second fundamental form tensor**) of S in $(\mathcal{V}, g)$:

$$\mathrm{II} : \mathfrak{X}(S) \times \mathfrak{X}(S) \longrightarrow \mathfrak{X}(S)^{\perp}$$

defined by

$$-\mathrm{II}(X,Y) \equiv (\nabla_X Y)^{\perp} \qquad \forall X, Y \in \mathfrak{X}(S)$$

- Observe that

$$\nabla_X Y = \overline{\nabla}_X Y - \mathrm{II}(X,Y) \qquad \forall X, Y \in \mathfrak{X}(S)$$

- II contains the information concerning the "shape" of S within $\mathcal{V}$ along *all* directions normal to S.

Definition (Totally geodesic S)

The submanifold S is called totally geodesic if every geodesic within $(S, \overline{g})$ is a geodesic of the space-time $(\mathcal{V}, g)$. Equivalently, if

$$\mathrm{II} = 0$$

Second fundamental forms

Definition (Second fundamental form relative to $N \in \mathfrak{X}(S)^{\perp}$)

$\forall N \in \mathfrak{X}(S)^{\perp}$, *the second fundamental form of S in $(\mathcal{V}, g)$ relative to N is the 2-covariant symmetric tensor field on S defined by*

$$K_N(X,Y) \equiv g\left(N, \mathbb{I}(X,Y)\right), \qquad \forall X, Y \in \mathfrak{X}(S).$$

The shape tensor decomposes as

$$\mathbb{I}(X,Y) = -K_k(X,Y)\,\ell - K_\ell(X,Y)\,k$$

in the null basis.

Observe that this formula is invariant under the boost freedom.

Weingarten operators

Definition (Weingarten operator relative to $N \in \mathfrak{X}(S)^\perp$)

The Weingarten operator $A_N : \mathfrak{X}(S) \longrightarrow \mathfrak{X}(S)$ associated to $N \in \mathfrak{X}(S)^\perp$ is defined by

$$A_N(X) \equiv (\nabla_X N)^T \qquad \forall X \in \mathfrak{X}(S)$$

Observe that $\nabla_X N = A_N(X) + D_X N$ and that

$$\overline{g}(A_N(X), Y) = K_N(X, Y), \quad \forall X, Y \in \mathfrak{X}(S)$$

From the symmetry of K_N it follows that, at each $x \in S$, $A_N|_x$ is a self-adjoint (with respect to $\overline{g}$) linear transformation on $T_x S$.

The mean curvature vector H

Definition (The mean curvature vector field H)

The mean curvature vector field $H \in \mathfrak{X}(S)^{\perp}$ is defined as the trace of the shape tensor with respect to $\overline{g}$:

$$H \equiv \sum_{i=1}^{n} \mathrm{II}(e_i, e_i)$$

for any ON basis $\{e_i\}$ in $\mathfrak{X}(S)$. Explicitly in a null basis

$$H \equiv -(\mathrm{tr}\, A_k)\, \ell - (\mathrm{tr}\, A_\ell)\, k$$

Notice that H and $\star^{\perp} H$ are well-defined, and their expressions in null bases are invariant under the boost gauge freedom.

Definition (Extremal (minimal) S)

The submanifold S is called stationary or extremal if

$$H = 0$$

The expansions

Definition (Expansion along $N \in \mathfrak{X}(S)^{\perp}$)

Each component of H along a particular normal direction

$$\theta_N \equiv g(H, N) = \mathrm{tr} A_N$$

is termed "expansion along N" of S. In particular,

$$\theta_k \equiv g(H, k) = \mathrm{tr} A_k, \qquad \theta_\ell \equiv g(H, \ell) = \mathrm{tr} A_\ell$$

are called the null expansions.

Important: Note that the expansions are not invariant under the boost freedom, e.g. $\theta_{k'} = e^{-\beta}\theta_k$ and $\theta_{\ell'} = e^{\beta}\theta_\ell$, however their signs are invariant.

Notice that $\star^{\perp} H$ is the (generically unique) normal direction with zero expansion: $\theta_{\star^{\perp} H} = \mathrm{tr} A_{\star^{\perp} H} = g(H, \star^{\perp} H) = 0$.

On the causal character of H

- Many important type of submanifolds in Gravitation are defined according to the causal orientation of H.
- For instance, if H is future timelike everywhere, then S is called a *future trapped submanifold*. Similarly to the past.
- *Marginally outer trappped surfaces* (MOTS) are characterized by H pointing along one of the normal null directions (ℓ or k) everywhere on S (therefore, H is null in this case).
- If in addition to S being a MOTS, H is future pointing everywhere, then S is called a *marginally future-trapped surface*.
- There are many other interesting cases...

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The total shear tensor $\tilde{\mathbb{I}}$

Definition (Total shear tensor)

The total shear tensor is defined by

$$\tilde{\mathbb{I}}(X,Y) \equiv \mathbb{I}(X,Y) - \frac{1}{n}\overline{g}(X,Y)H \quad \forall X,Y \in \mathfrak{X}(S)$$

- $\tilde{\mathbb{I}}$ gives the **trace-free part** (with respect to $\overline{g}$) of the shape tensor $\mathbb{I}$
- $\tilde{\mathbb{I}}$ contains the information concerning the "deformation" of S within $\mathcal{V}$, **while keeping its volume fixed**, along *all* directions normal to S.

Definition (Totally umbilical S)

The submanifold S is called totally umbilical if the shape tensor is proportional to the first fundamental form. Equivalently, if

$$\tilde{\mathbb{I}} = 0$$

Shear tensors

Definition (Shear tensor relative to $N \in \mathfrak{X}(S)^\perp$)

$\forall N \in \mathfrak{X}(S)^\perp$, the shear tensor of S in $(\mathcal{V}, g)$ relative to N is the 2-covariant symmetric tensor field on S defined by

$$\widetilde{K}_N(X,Y) \equiv g\left(N, \tilde{\mathbb{I}}(X,Y)\right), \qquad \forall X, Y \in \mathfrak{X}(S)\,.$$

The total shear decomposes as

$$\tilde{\mathbb{I}}(X,Y) = -\widetilde{K}_k(X,Y)\,\ell - \widetilde{K}_\ell(X,Y)\,k$$

in the null basis.

Observe again that this formula is invariant under the boost freedom.

Shear operators

Definition (Shear operator relative to $N \in \mathfrak{X}(S)^{\perp}$)

The *shear operator* $\widetilde{A}_N : \mathfrak{X}(S) \longrightarrow \mathfrak{X}(S)$ *associated to* $N \in \mathfrak{X}(S)^{\perp}$ *is defined by*

$$\overline{g}(\widetilde{A}_N(X), Y) = \widetilde{K}_N(X, Y), \quad \forall X, Y \in \mathfrak{X}(S)$$

From the symmetry of $\widetilde{K}_N$ it follows that, at each $x \in S$, $\widetilde{A}_N|_x$ is a self-adjoint (with respect to $\overline{g}$) linear transformation on T_xS.

1. $\widetilde{A}_N$ is the trace-free part of the Weingarten operator A_N

$$\widetilde{A}_N = A_N - \frac{1}{n}\mathrm{tr}A_N\mathbf{1} = A_N - \frac{1}{n}\theta_N\mathbf{1}$$

2. Similarly

$$\widetilde{K}_N = K_N - \frac{1}{n}\theta_N\,\overline{g}$$

Shear scalars

Definition (Shear scalar relative to $N \in \mathfrak{X}(S)^\perp$)

The shear scalar σ_N *associated to* $N \in \mathfrak{X}(S)^\perp$ *is defined by*

$$\sigma_N^2 = tr(\widetilde{A}_N)^2$$

One can easily check that σ_N^2 is non-negative, and actually

$$\sigma_N = 0 \iff \widetilde{A}_N = 0.$$

σ_k and σ_ℓ are called the *null shear scalars*.

A scalar product for self-adjoint operators

- Let $\mathfrak{T}(S)$ be the set of self-adjoint operators on S (i.e. (1,1)-tensor fields on S which are self-adjoint with respect to $\overline{g}$ at each point $x \in S$).
- One can define a scalar product $\langle \, , \, \rangle$ on $\mathfrak{T}(S)$ by means of

$$\langle A, B \rangle \equiv \mathrm{tr}(AB), \qquad \forall A, B \in \mathfrak{T}(S)$$

- This scalar product is symmetric and **positive definite** (at each $x \in S$).
- Observe that, for each shear operator $\widetilde{A}_N$, we have

$$\sigma_N^2 = \left\langle \widetilde{A}_N, \widetilde{A}_N \right\rangle$$

Umbilical-type submanifolds

Definition (N-Umbilical submanifolds)

S is said to be *umbilical along a vector field* $N \in \mathfrak{X}(S)^\perp$ —(or simply N-umbilical)— if the corresponding Weingarten operator is proportional to the Identity

$$A_N = \frac{1}{n} g(H, N) \mathbf{1}$$

or equivalently, if $K_N = \frac{1}{n} g(H, N)\, \bar{g}$.

Observe that S is N-umbilical if and only if

$$\widetilde{A}_N = 0$$

Therefore, if and only if

$$\sigma_N = 0$$

Pseudo-umbilical and Ortho-umbilical S

Definition (Pseudo-umbilical S)

S is said to be pseudo-umbilical if it is umbilical with respect to $N = H$, so that $\widetilde{A}_H = 0$

Definition (Ortho-umbilical)

And S will be called ortho-umbilical if it is umbilical with respect to $N = \star^{\perp} H$, so that $\widetilde{A}_{\star^{\perp} H} = 0$ $(= A_{\star^{\perp} H})$.

A submanifold can be pseudo- and ortho-umbilical at the same time.
This requires that H be null necessarily: $g(H, H) = 0$.
Thus, such S include the umbilical MOTS.

A first result

Proposition (Characterization of ortho-umbilical S)

There exists $N \in \mathfrak{X}(S)^{\perp}$ such that $K_N = 0$ if and only if S is ortho-umbilical (and thus, necessarily N points along $\star^{\perp} H$).

This implies that the ortho-umbilical case is somewhat exceptional.

Note that this is equivalent to stating that $\mathbb{I}(X, Y)$ points along H for any $X, Y \in \mathfrak{X}(S)$.

The first Main Theorem

Theorem

Let S be a co-dimension two spacelike submanifold. The following conditions are equivalent:

1. S *is umbilical along a vector field* $N \in \mathfrak{X}(S)^\perp$
2. *The total shear satisfies* $\tilde{\mathrm{II}}(X,Y)^\flat \wedge \tilde{\mathrm{II}}(Z,W)^\flat = 0$ $\forall X, Y, Z, W \in \mathfrak{X}(S)$
3. *The null shear tensors* $\widetilde{K}_\ell$ *and* $\widetilde{K}_k$ *are such that*
$$\widetilde{K}_\ell \otimes \widetilde{K}_k = \widetilde{K}_k \otimes \widetilde{K}_\ell$$
4. *Any two shear tensors satisfy*
$$\widetilde{K}_M \otimes \widetilde{K}_N = \widetilde{K}_N \otimes \widetilde{K}_M \quad \forall M, N \in \mathfrak{X}(S)^\perp$$
5. *For* $M, N \in \mathfrak{X}(S)^\perp$*:* $\left\langle \widetilde{A}_M, \widetilde{A}_N \right\rangle^2 = \sigma_N^2 \sigma_M^2$

Remarks

- The last condition is very friendly from an operational point of view. If one wishes to know whether or not an S has an umbilical normal direction, one just computes two shear operators (one can choose the easiest ones) and then it is a question of multiplying the matrices and taking traces.
- Note that the 3rd and 4th conditions can be equally written using shear operators:

$$\widetilde{A}_M \otimes \widetilde{A}_N = \widetilde{A}_N \otimes \widetilde{A}_M$$

- This implies in particular that any two shear operators, and any two Weingarten operators, commute

$$[\widetilde{A}_M, \widetilde{A}_N] = 0 = [A_M, A_N]$$

  which provides another very simple test for obstruction to the existence of an umbilical direction.
- Actually, this last condition is also sufficient in $n = 2$, that is, in 4-dimensional spacetimes.

The main theorem for surfaces: $n = 2$

Theorem (Umbilical surfaces (Senovilla 2011))

In 4-dimensional spacetimes, the necessary and sufficient condition for a surface S to be umbilical along a normal direction is that two independent Weingarten operators commute.

Then, actually all possible Weingarten operators do commute.

This is equivalent to the condition that the shape tensor $\mathbb{I}$ be diagonalizable on S.

Hence, there exists a (generally unique) ON basis on S in which all possible Weingarten operators diagonalize.

A new vector field G

- The main Theorem implies that whenever there is an umbilical direction, the total shear $\tilde{\mathbb{I}}$ spans a single normal direction
- That is, there exists a vector field $G \in \mathfrak{X}(S)^{\perp}$ such that

The vector field G

$$\tilde{\mathbb{I}}(X,Y) = \widetilde{K}(X,Y)G \quad \forall X,Y \in \mathfrak{X}(S)$$

where $\widetilde{K}$ is a symmetric 2-covariant tensor field in S

- If $\widetilde{A}$ denotes the self-adjoint operator associated to $\widetilde{K}$, that is $g(\widetilde{A}(X),Y) = \widetilde{K}(X,Y)$ for all $X,Y \in \mathfrak{X}(S)$, then the following properties hold

1. $\mathrm{tr}\widetilde{A} = 0$
2. One can choose a normalization such that $\left\langle \widetilde{A}, \widetilde{A} \right\rangle = 1$
3. Obviously, this entails $\widetilde{A} \neq 0$ and fixes G (up to sign)
4. $\Rightarrow g(G,N) = \sigma_N, \ \forall N \in \mathfrak{X}(S)^{\perp}$
5. $\widetilde{K}_N = \sigma_N \widetilde{K}, \quad \widetilde{A}_N = \sigma_N \widetilde{A}, \quad \forall N \in \mathfrak{X}(S)^{\perp}$

The second main theorem

Theorem (Uniqueness of the umbilical direction)

The umbilical direction, if it exists, is unique and given by $\star^{\perp} G$ *—unless, of course,* $G = 0$ *which characterizes the totally umbilical case.*

Observe that $G = -\sigma_k \, \ell - \sigma_\ell \, k$

Thus, the umbilical direction is given by

$$\star^{\perp} G = -\sigma_k \, \ell + \sigma_\ell \, k$$

Curvatures

- The intrinsic curvature for $(S, \bar{g})$ has the usual definition

$$\overline{R}(X,Y)Z \equiv \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z, \quad \forall X, Y, Z \in \mathfrak{X}(S)$$

- Similarly, the normal curvature is defined on S by

$$R^{\perp}(X,Y)N \equiv D_X D_Y N - D_Y D_X N - D_{[X,Y]} N, \quad \forall N \in \mathfrak{X}(S)^{\perp}$$

- A simple calculation provides

$$R^{\perp}(X,Y)N = ds(X,Y) \star^{\perp} N$$

for all $X, Y \in \mathfrak{X}(S)$ and for all $N \in \mathfrak{X}(S)^{\perp}$. Thus, we justify that s characterizes the normal connection and that ds defines its curvature.

Another theorem

Theorem

If S is umbilical along a normal direction then

$$R^{\perp}(X,Y)N = (R(X,Y)N)^{\perp}\,, \qquad \forall X,Y\,\mathfrak{X}(S),\;\; \forall N \in \mathfrak{X}(S)^{\perp}$$

This is yet equivalent to $\forall X,Y\,\mathfrak{X}(S),\;\; \forall N,M \in \mathfrak{X}(S)^{\perp}$

$$g(M, R(X,Y)N) = ds(X,Y)\; g(\star^{\perp}N, M).$$

This follows easily from the Ricci equation

$$g(M, R(X,Y)N) = g\left([A_M, A_N](Y), X\right) + d\omega(X,Y)\; g(\star^{\perp}N, M)$$

Actually, this condition is also **sufficient** when $n = 2$
(4-dimensional spacetimes)

An interesting Corollary

Corollary

In locally conformally flat spacetimes (including Lorentz space forms) if a co-dimension two submanifold is umbilical along a normal direction then its normal curvature vanishes:

$$R^{\perp} = 0$$

It is easily checked that $(R(X,Y)N)^{\perp} = (W(X,Y)N)^{\perp}$ where W denotes de Weyl conformal curvature.

Consequently, if $(\mathcal{V}, g)$ is locally conformally flat, then $(R(X,Y)N)^{\perp} = 0$ for all $X, Y \in \mathfrak{X}(S)$ and for all $N \in \mathfrak{X}(S)^{\perp}$, and the result follows immediately from the previous theorem. Again, this condition is also **sufficient** when $n = 2$ (4-dimensional spacetimes)

The Casorati operator and curvature

- The Casorati operator $B : \mathfrak{X}(S) \to \mathfrak{X}(S)$ can be defined as

$$B \equiv \frac{1}{g(N,N)} \left(A_N^2 - A^2_{\star^\perp N} \right)$$

for any **non-null** $N \in \mathfrak{X}(S)^\perp$.

- This definition is independent of the choice of N
- One can check that $B = -\{A_k, A_\ell\}$, using null bases
- The Casorati curvature is defined by tr$B = -2 \langle A_k, A_\ell \rangle$
- Similarly, we can define another operator $J : \mathfrak{X}(S) \to \mathfrak{X}(S)$ and associated curvature tr$J$ using shear objects

$$J = -\left\{ \widetilde{A}_k, \widetilde{A}_\ell \right\}, \qquad \mathrm{tr} J = -2 \left\langle \widetilde{A}_k, \widetilde{A}_\ell \right\rangle$$

- It is easily proven that

$$B - J = \frac{2}{n} \widetilde{A}_H + \frac{1}{n^2} g(H,H) \mathbf{1}$$

- Therefore tr$B -$ tr$J = g(H,H)/n$.

Causal character of the umbilical direction

If an umbilical direction exists, a criteria for determining its causal character is provided by the sign of tr$J$:

$$\text{tr}J = -2\left\langle \widetilde{A}_k, \widetilde{A}_\ell \right\rangle = \begin{cases} > 0 \Rightarrow \star^\perp G & \text{is timelike} \\ < 0 \Rightarrow \star^\perp G & \text{is spacelike} \\ = 0 \Rightarrow \star^\perp G & \text{is lightlike} \end{cases}$$

Corollary: Pseudo-umbilical surfaces

Corollary (Pseudo-umbilical characterization)

A (non-minimal) co-dimension two spacelike submanifold S is pseudo-umbilical if and only if $B - J$ is proportional to the Identity.

Corollary (Pseudo- and ortho-umbilical)

A (non-minimal and non-totally umbilical) co-dimension two spacelike submanifold S is pseudo-umbilical as well as ortho-umbilical if and only if

$$g(H, H) = 0, \qquad B = 0 = J$$

Examples: Conformal Killing vector ξ

Example (Integrable conformal Killing vector field)

In arbitrary spacetimes, let ξ be a conformal Killing vector

$$\pounds_\xi g = 2\phi g$$

that generates orthogonal hypersurfaces: $\xi^\flat \wedge d\xi^\flat = 0$.
Then any spacelike co-dimension 2 submanifold imbedded in any of its orthogonal hypersurfaces is ξ-umbilical.

Proof.

$$\forall X, Y \in \mathfrak{X}(S), \qquad 2\phi\, g(X,Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X)$$
$$= -g(A_\xi(X), Y) - g(A_\xi(Y), X)$$
$$= -2g(A_\xi(X), Y)$$

ergo

$$A_\xi = -\phi\,\mathbf{1}.$$

Q.E.D.

Examples: Kerr Black Hole

Example (The Kerr spacetime)

In the so-called advanced Eddington-Finkelstein coordinates $\{v, r, \theta, \varphi\}$ *the line-element of the Kerr space-time reads*

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2mr}{\rho^2}\right) dv^2 + 2\ dv\ dr + \rho^2 d\theta^2 \\ &\quad - \frac{4amr\sin^2\theta}{\rho^2} d\varphi\ dv - 2a\sin^2\theta\ d\varphi\ dr \\ &\quad + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta d\varphi^2, \end{aligned}$$

where

$$\rho^2 \equiv r^2 + a^2\cos^2\theta$$

and

$$\Delta = \Delta(r) \equiv r^2 - 2rm + a^2$$

Examples: Kerr Black Hole (cont'd)

- This spacetime represents a rotating black hole
- m represents the mass of the black hole
- a describes the angular momentum per unit mass of the black hole
- $a = 0$ leads to the Schwarzschild black hole with mass M.
- The space-time is axially symmetric, ∂_φ being a rotational Killing vector, and stationary for $\Delta > 0$.
- There is a family of preferred surfaces defined by constant values of v and r:

$$v = \bar{v}, \qquad r = \bar{r}\,.$$

- These are compact, topologically S^2.
- One can easily compute their 1st fundamental forms
- Two convenient normal 1-forms for these surfaces are given by

$$N^\flat = dv, \qquad M^\flat = dr$$

Examples: Kerr Black Hole (cont'd)

- The corresponding second fundamental forms are easily computed (in the natural basis $\{d\theta, d\varphi\}$, and letting $r = \bar{r}$ everywhere):

$$K_M = -\frac{\Delta}{\rho^2}\begin{pmatrix} r & 0 \\ 0 & \left(r + \frac{m}{\rho^4}a^2(a^2\cos^2\theta - r^2)\sin^2\theta\right)\sin^2\theta \end{pmatrix}$$

- $K_N =$

$$\begin{pmatrix} -\frac{r}{\rho^2}(r^2+a^2) & 2\frac{m}{\rho^4}ra^3\sin^3\theta\cos\theta \\ 2\frac{m}{\rho^4}ra^3\sin^3\theta\cos\theta & -\frac{r^2+a^2}{\rho^2}\left(r + \frac{m}{\rho^4}a^2(a^2\cos^2\theta - r^2)\sin^2\theta\right)\sin^2\theta \end{pmatrix}$$

- Then, it is straightforward to check when the Weingarten operators commute $[A_M, A_N] = 0$ (i.e., when there exists an umbilical direction):

$$\Longleftrightarrow \qquad \text{either } \theta = 0, \frac{\pi}{2}, \pi \text{ or } r = 0, \text{ or } \Delta = 0$$

Examples: Kerr Black Hole (cont'd)

- $\theta = 0, \pi$ are the intersection of these surfaces with the axis of symmetry (north and south poles).
- $\theta = \pi/2$ is the equatorial plane for these surfaces: notice that there is equatorial symmetry in the space-time
- The surfaces with $r = 0$ are not complete nor compact, because they contain the curvature singularity of the space-time at their equatorial ring.
- Finally, $\Delta = 0$ defines the Killing horizon of this space-time, which corresponds to the Event Horizon too.
- This horizon is a null hypersuface foliated by compact surfaces that are umbilical along the null generator
- Actually, as follows from the previous expression, $K_M = 0$, and thus these are pseudo- and ortho-umbilical surfaces!
- This is a general property of Non-Expanding Horizons (including Isolated and Killing Horizons).

Final considerations: H versus G

H always exists

$H = 0$ stationary/extremal

$H \neq 0 \Rightarrow \star^{\perp} H$ is the unique direction with zero expansion

$g(H, N) =$ expansion along N $(= \theta_N)$

Causal character of H determines trapping, or untrapping, etc.

H arises in the variational problem for the volume of S

G requires condition on A_N's

$G = 0$ totally umbilical

$G \neq 0 \Rightarrow \star^{\perp} G$ is the unique umbilical direction

$g(G, N) =$ shear scalar along N $(= \sigma_N)$

Causal character of G determines the curvature tr$J$

Variational problem involving G? (Shape for fixed volume?)

Thank you.   Obrigado

REFERENCES

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