Unicity of certain solutions of discrete Painlevé II

Walter Van Assche

KU Leuven, Belgium

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Discrete Painlevé II

Discrete Painlevé II (d-P_{II}) is

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We will only consider the case $\beta = 0 = \gamma$

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Motivation 1

Orthogonal polynomials on the unit circle

$$\frac{1}{2\pi}\int_0^{2\pi} \Phi_n(z)\overline{\Phi_m(z)}e^{\lambda\cos\theta}\,d\theta = 0, \qquad m \neq n, z = e^{i\theta}$$

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$$\Phi_{n+1}(z) = z\Phi_n(z) - \alpha_n\Phi_n^*(z),$$

If $\alpha_n = x_{n+1}$, then

$$x_{n+1} + x_{n-1} = \frac{-2nx_n}{\lambda(1-x_n^2)}, \qquad x_0 = -1, x_1 = \frac{I_1(\lambda)}{I_0(\lambda)}.$$

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modified Bessel function

$$I_n(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{n+2k}}{\Gamma(\nu+k+1)k!} = \frac{1}{\pi}\int_0^{\pi} e^{x\cos\theta}\cos(n\theta)\,d\theta.$$

Motivation 2

Generalized Charlier polynomials

$$\sum_{k=0}^{\infty} P_n(k)P_m(k)\frac{a^k}{(k!)^2} = 0, \qquad m \neq n, a > 0$$

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$$xP_n(x) = P_{n+1}(x) + b_nP_n(x) + a_n^2P_{n-1}(x),$$

$a_n^2 = a(1 - c_n^2)$, $b_n = n + \sqrt{a}c_nc_{n+1}$

$$c_{n+1} + c_{n-1} = \frac{nc_n}{\sqrt{a}(1 - c_n^2)}, \qquad c_0 = 1, c_1 = \frac{I_1(2\sqrt{a})}{I_0(2\sqrt{a})}.$$

Main theorem

Theorem

Suppose that $\alpha > 0$. There is a unique solution of

$$x_{n+1} + x_{n-1} = \frac{\alpha n x_n}{1 - x_n^2}$$

which satisfies $x_0 = 1$ and $-1 < x_n < 1$ for $n \geq 1$. This solution corresponds to the initial condition $x_1 = I_1(2/\alpha)/I_0(2/\alpha)$ and is positive for every $n \geq 0$.

Corollary

Suppose that $\alpha > 0$. There is a unique solution of

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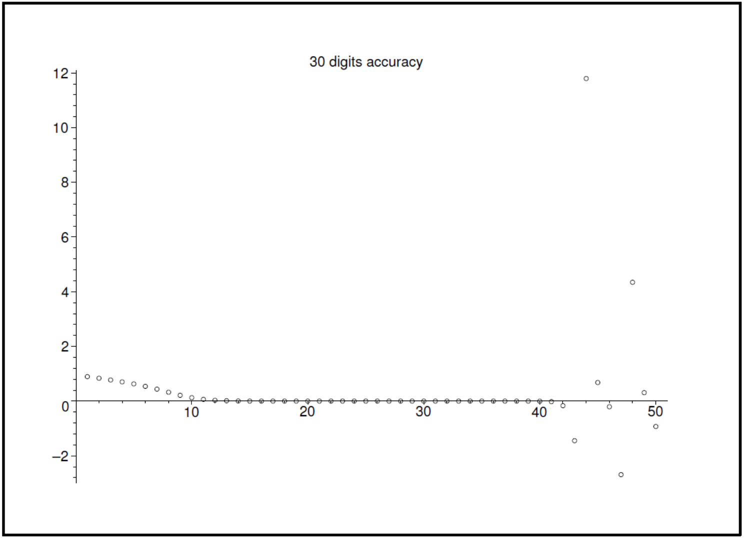
30 digits accuracy
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40
50

Proof when $\alpha > 1$

Solve

$$x_{n+1} + x_{n-1} = \frac{\alpha n x_n}{1 - x_n^2}$$

for x_n, then

$$x_n = \frac{-\alpha n \pm \sqrt{(\alpha n)^2 + 4(x_{n+1} + x_{n-1})}}{2(x_{n+1} + x_{n-1})}.$$

We will only take the solution with the $+$ sign.

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$$x_n = f\left(\frac{2(x_{n+1} + x_{n-1})}{\alpha n}\right), \qquad n \geq 1,$$

$$f(t) = \frac{-1 + \sqrt{1 + t^2}}{t} = \frac{t}{1 + \sqrt{1 + t^2}}, \qquad t \in \mathbb{R}.$$

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Note that $-1 < f(t) < 1$ and $f'(t) \leq \frac{1}{2}$.

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Operator $F = S \to S$

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$$\|Fx - Fy\| \leq \frac{1}{\alpha}\|x - y\|.$$

hence F is a contraction on S and there is a unique fixed point which satisfies d-P_{II}.

Proof for all $\alpha > 0$

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$$x0 = (1,0,0,0,\ldots) \in S_+ \Rightarrow Fx0 = (1, f(2/\alpha), 0, 0, \ldots) \geq x0$$

$$F^n x0 \geq F^{n-1} x0 \geq x0, \qquad n \geq 1$$

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$$x_- = \lim_{n\to\infty} F^n x0, \quad x_+ = \lim_{n\to\infty} F^n x_1 \in S_+$$

x_- and x_+ are fixed points of F and for every fixed point $x_f \in S_+$

$$x_- \le x_f \le x_+$$

Unique fixed point?

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Property

$$r_{n+1} = r_1 \det \begin{pmatrix} \alpha c_1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 2\alpha c_2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 3\alpha c_3 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & \ddots & 1 \\ 0 & \cdots & 0 & 0 & 1 & n\alpha c_n \end{pmatrix} = r_1 q_{n+1},$$

where $n\alpha c_n > 2$ *and* $\lim_{n\to\infty} c_n = 1$.

Behavior of r_n

Theorem

The only solution r with $r_0 = 0$ and $0 \leq r_n \leq 1$ for $n \geq 1$ is the zero solution $r_n = 0$.

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If $r_{n+1} = r_1 q_n$ and $q_n = \alpha^n n! c_1 c_2 \cdots c_n \, p_n$, then

$$p_n - p_{n-1} = -\frac{p_{n-2}}{\alpha^2 n(n-1) c_n c_{n-1}}, \qquad p_0 = 1 = p_1.$$

$(p_n)_n$ is a decreasing sequence.

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1. if $p_n \to p > 0$ then $q_n \to +\infty$, hence $r_n \leq 1$ only if $r_1 = 0$, and hence $r_n = 0$ for all n.

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2. $p_n \to 0$ cannot happen

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$$p_{n+1} = \det \begin{pmatrix} 1 & a_1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & a_2 & 0 & \cdots & 0 \\ 0 & a_2 & 1 & a_3 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{n-1} & 1 & a_n \\ 0 & \cdots & 0 & 0 & a_n & 1 \end{pmatrix}$$

where $a_n = \frac{1}{\alpha\sqrt{n(n+1)c_n c_{n+1}}}$, *with* $n\alpha c_n > 2$ *and* $\lim_{n\to\infty} c_n = 1$.

Some operator theory

The infinite matrix

$$A = \begin{pmatrix} 1 & a_1 & 0 & 0 & \cdots & 0 & & \\ a_1 & 1 & a_2 & 0 & \cdots & 0 & 0 & \\ 0 & a_2 & 1 & a_3 & \cdots & 0 & 0 & \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 & \\ 0 & \cdots & 0 & a_{n-1} & 1 & a_n & 0 & \\ 0 & \cdots & 0 & 0 & a_n & 1 & a_{n+1} & \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots \end{pmatrix}$$

is a compact perturbation of the identity matrix.

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- $p_n \to 0$ if and only if $0 \in \sigma(A)$.

Bounds

Theorem

The unique solution of d-P_{II} with $x_0 = 1$ and $0 < x_n < 1$ satisfies

$$x_n \leq \frac{2^n}{\alpha^n} \frac{1}{(2n-1)!!} = \frac{4^n}{\alpha^n} \frac{n!}{(2n)!}, \qquad n \geq 0.$$

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$$x_n \leq b_{n,n}.$$

Ablowitz-Ladik lattice

Theorem

Let μ be a positive measure on the unit circle which is symmetric (real reflection/Verblunsky coefficients). Let μ_t be the measure such that $d\mu_t(\theta) = e^{t\cos\theta}\, d\mu(\theta)$, with $t \in \mathbb{R}$. Then the reflection/Verblunsky coefficients of μ_t satisfy

$$2\frac{d}{dt}\alpha_n = (1-\alpha_n^2)(\alpha_{n+1}-\alpha_{n-1}), \qquad n \geq 0.$$

Painlevé V

$$\alpha_{n+1} + \alpha_{n-1} = \frac{-2(n+1)\alpha_n}{t(1-\alpha_n^2)} \qquad \text{d-P}_{\text{II}}$$

$$\alpha_{n+1} - \alpha_{n-1} = \frac{2\alpha_n'}{1-\alpha_n^2}, \qquad \text{A-L lattice}$$

leads to

$$\alpha_n'' = \frac{-\alpha_n}{1-\alpha_n^2}(\alpha_n')^2 - \frac{\alpha_n'}{t} - \alpha_n(1-\alpha_n^2) + \frac{(n+1)^2}{t^2}\frac{\alpha_n}{1-\alpha_n^2}.$$

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If

$$\alpha_n = \frac{1+y}{1-y}$$

then y satisfies **Painlevé V**.

Relationship between solutions of Painlevé III and Painlevé V.

Property

If $w(x; a, b, 1, -1)$ *is a solution of Painlevé III, and*

$$v = w' - \epsilon w^2 + \frac{(1 - \epsilon a)w}{x}$$

then $\frac{v-1}{v+1}$ *satisfies Painlevé V.*

Painlevé V and Painlevé III

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- special solution of **Painlevé III** in terms of modified Bessel functions I_n and K_n. (i.e, when $\epsilon_1 a + \epsilon_2 b = 4k + 2$)

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Relationship between solutions of Painlevé III and Painlevé V.

Property

If $w(x; a, b, 1, -1)$ *is a solution of Painlevé III, and*

$$v = w' - \epsilon w^2 + \frac{(1 - \epsilon a)w}{x}$$

then $\frac{v-1}{v+1}$ *satisfies Painlevé V.*

- special solution of **Painlevé III** in terms of modified Bessel functions I_n and K_n. (i.e, when $\epsilon_1 a + \epsilon_2 b = 4k + 2$)
- The positivity of the solution corresponds to the solution containing only I_n.

Painlevé V and Painlevé III

Relationship between solutions of Painlevé III and Painlevé V.

Property

If $w(x; a, b, 1, -1)$ is a solution of Painlevé III, and

$$v = w' - \epsilon w^2 + \frac{(1 - \epsilon a)w}{x}$$

then $\frac{v-1}{v+1}$ satisfies Painlevé V.

- special solution of **Painlevé III** in terms of modified Bessel functions I_n and K_n. (i.e, when $\epsilon_1 a + \epsilon_2 b = 4k + 2$)
- The positivity of the solution corresponds to the solution containing only I_n.
- x_n has an explicit expression using Wronskian determinants containing the modified Bessel functions $I_0, \ldots, I_n$.

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